Mathematisches Institut
der Universität München
Pascal Reisert

Summer Term 2016
Problem sheet 5
May 18th, 2016

Riemann Surfaces
Solution

Problem 18

b) There is no two-sheeted cover of E_ϱ onto itself: Let ω_1, ω_2 be linear independent of smallest absolute value in $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $|\omega_1| \leqslant |\omega_2|$.
By problem 17, there should be a constant α with $\alpha\Lambda \subset \Lambda$ and suitable index, i.e. $[\omega_1] = [\omega_2] \neq [0] = [\omega_1 + \omega_2] = \mod \alpha\Lambda$.[1] The two conditions imply $\alpha\omega_1 = k_{11}\omega_1 + k_{12}\omega_2, \alpha\omega_2 = k_{21}\omega_1 + k_{22}\omega_2$ and $\omega_1 - \omega_2 = +k_{-,1}\alpha\omega_1 + k_{-,2}\alpha\omega_2$ and $\omega_1 + \omega_2 = \alpha(k_{+,1}\omega_1 + k_{+,2}\omega_2)$ for some integers $k_{11}, k_{12}, k_{21}, k_{22}, k_{-,1}, k_{-,2}, k_{+,1}, k_{+,2}$. We claim that $\alpha\omega_1 = \pm\omega_2$ or $\alpha\omega_1 = \pm(\omega_1 \pm \omega_2)$. Note that $\alpha\omega_1 = k\omega_1 \Rightarrow \alpha = k \in \mathbb{Z}$ and the contradiction $[0] \neq [\omega_1] \neq [\omega_2] \neq [0] \mod k\Lambda$ if $|k| > 1$ and $\alpha\Lambda = \Lambda$ for $|k| = 1$ follows.[2] Now assume $\alpha\omega_1 \neq \pm\omega_2, \mathbb{Z}\omega_1$, then $|\alpha\omega_1|$ at least as large as the smallest absultute value in $\Lambda \setminus \{\mathbb{Z}\omega_1, \pm\omega_2\}$, i.e. is at least $|\omega_1 \pm \omega_2|$.[3] Then

$$|\omega_1 - \omega_2| = |k_{-,1}\alpha\omega_1 + k_{-,2}\alpha\omega_2| \geqslant |\alpha\omega_1|$$

and

$$|\omega_1 + \omega_2| = |k_{+,1}\alpha\omega_1 + k_{+,2}\alpha\omega_2| \geqslant |\alpha\omega_1|$$

implies $\alpha\omega_1 = \pm(\omega_1 \pm \omega_2)$. In the case $\omega_1 = 1, \omega_2 = \varrho = \exp(2\pi i/3)$ we have $|\omega_1| = |\omega_2| = |\omega_2 - \omega_1| = |\exp(\pi i/3)| = 1$ and we get $\alpha = \exp(ki\pi/3), k \in \mathbb{Z}$ and hence $\alpha\Lambda = \Lambda$.

Alternative 1: The following proof was given by Ludwig Fürst. Note that $1 \equiv \varrho \equiv 1 + \varrho \mod \alpha\Lambda$ implies $1, \varrho \in \alpha\Lambda$ and hence $\alpha\Lambda = \Lambda$. Furthermore since $1 + \varrho + \varrho^2 = 0$ we know that for all $\omega \in \Lambda, \omega\alpha\Lambda \subset \alpha\Lambda$. Hence if

(i) $1 \in \alpha\Lambda$ then $\varrho \cdot 1 = \varrho \in \alpha\Lambda \Rightarrow \alpha\Lambda = \Lambda$.
(ii) $\varrho \in \alpha\Lambda$ then $(1 + \varrho) \cdot \varrho = \varrho^2 + \varrho = -1 \Rightarrow \alpha\Lambda = \Lambda$.
(iii) $1 + \varrho \in \alpha\Lambda$ then $\varrho \cdot (\varrho + 1) = \varrho^2 + \varrho = -1 \Rightarrow \alpha\Lambda = \Lambda$.

There can be no two-sheeted covering $E_\varrho \to E_\varrho$.

Alternative 2: A number theoretic proof by Alexander Trost.
Λ_ϱ forms the ring of integers $\mathcal{O}_K$ for the number field $K = \mathbb{Q}[\sqrt{-3}]$. We are looking for a α with $\alpha\Lambda_\varrho \subset \Lambda_\varrho$. Observe that $\alpha\Lambda_\varrho$ is a prinicpal ideal that contains $\alpha \cdot 1$. We require $2 = [\Lambda_\varrho : \alpha\Lambda_\varrho] = N(\alpha\Lambda_\varrho) = N_{K|\mathbb{Q}}(\alpha) = \alpha \cdot \overline{\alpha}$.[4] But

$$2 = (k + l\varrho)(k + l\overline{\varrho}) = k^2 + l^2 + (\varrho + \overline{\varrho})kl = k^2 + l^2 - kl = (k - l)^2 + kl$$

is impossible. More precisely if $kl < 0$ then $2 = k^2 + l^2 - kl$ has no integer solution; if $kl \geqslant 0$, then $2 = (k - l)^2 + kl$ has no integer solution. Hence we will find no suitable α.

[1] If $[\omega_1] = [0] \neq [\omega_2]$ substitute $\omega_1 \mapsto \omega_1 + \omega_2$.
[2] The coefficient of ω_2 in $\omega_1 = \omega_2 + k''k\omega_1 + l''k\omega_2$ is 1 mod k.
[3] Write $k\omega_1 + l\omega_2$ as $(k+l)(w_1+w_2)/2 - (k-l)(w_1-w_2)/2$, w.l.o.g. $|k+l| \geqslant |k-l|$ and estimate for $0 \neq k \neq l \neq 0$: $|k\omega_1 + l\omega_2| \geqslant |(k+l)/2||w_1+w_2| - |(k-l)/2||w_1-w_2| \geqslant |(k+l)/2||w_1+w_2| - |(k-l)/2||w_1-w_2| \geqslant \min\{k,l\}|w_1-w_2|$ for w.l.o.g. $|w_1+w_2| \geqslant |w_1-w_2|$.
[4] Recall that the norm is just $(k + l\varrho) \cdot (k + l\overline{\varrho})$.

Problem 20

a) $f = w + z = \sqrt[3]{1 - z^3} + z$ has no pole outside of ∞. We get no zero since $w = -z \Rightarrow w^3 + z^3 = 0 \neq 1$. Rewrite $z(\sqrt[3]{z^{-3} - 1} + 1)$. $\sqrt[3]{z^{-3} - 1}$ has three branches around ∞. We develop around $z = \infty$ resp. $u = 0, u = z^{-1}$

$$\sqrt[3]{u^3 - 1} = \exp(2\pi ik/3)(-1 + u^3/3 + \mathcal{O}(u^6)).$$

For $k = 0$ we get $z(\sqrt[3]{z^{-3} - 1} + 1) = z^{-2}/3 + \mathcal{O}(z^{-5})$. In the local coordinate $u = z^{-1}$ we have $u^2(1/3 + \mathcal{O}(u^3))$ and a zero of order 2. For $k = 1, 2$ we have $z(\sqrt[3]{z^{-3} - 1} + 1) = z(-1 + \exp(2\pi ik/3) + z^{-3}\exp(2\pi ik/3)/3 + \mathcal{O}(z^{-6}))$ and hence two single poles.
The two-sheeted cover has three branch points apart from 0: $w + z = c \in \mathbb{C}$, $w = c - z \Rightarrow (c - z)^3 + z^3 = 1 \Rightarrow c^3 - 1 - 3c^2 z + 3cz^2 = 0$. We get one-solution $z = -c/2$ for $9c^4 - 12c^4 + 12c = 3c(4 - c^3) = 0$ or $c = 0, \exp(2\pi ik/3)\sqrt[3]{4}$. Hence the branch points are $(-c/2, 3c/2)$.

b) Observer that $q = 1 + \exp(2\pi in/3) + \exp(4\pi in/3) = 0$. Locally $f_n = \exp(2\pi in/3)r(z) \cdot z + z, r(z) = \sqrt[3]{z^{-3} - 1}, n = 1, 2, 3$. Hence $s_0(f_1, f_2, f_3) = 1$, $s_1(f_1, f_2, f_3) = f_1 + f_2 + f_3 = 3z$,

$$s_2(f_1, f_2, f_3) = f_1 f_2 + f_1 f_3 + f_2 f_3 = z^2(3 + 2r(z)(q) + r(z)^2 q) = 3z^2$$

and

$$s_3(f_1, f_2, f_3) = f_1 f_2 f_3 = z^3(1 + r(z)q + r(z)^2 q + r(z)^3) = z^3(1 + z^{-3} - 1) = 1.$$