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Problem sheet 4
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Riemann Surfaces
Solution

Problem 15

a) Let z be a root, i.e. $F(z) = z^n + a_1 z^{n-1} + \ldots + a_n = 0$. We get $|z^n| = |\sum_{i=0}^{n-1} a_{n-i} z^i| \leqslant \sum_{i=0}^{n-1} |a_{n-i}||z|^i$ and $1 \leqslant \sum_{i=0}^{n-1} |a_{n-i}||z|^{i-n}$. Now if $|z| > 2\max\{|a_k|^{1/k} : 1 \leqslant k \leqslant n\}$ then

$$\sum_{i=0}^{n-1} |a_{n-i}||z|^{i-n} \leqslant \sum_{i=0}^{n-1} 2^{i-n} \cdot |a_{n-i}||a_{n-i}|^{(i-n)/(n-i)} = \sum_{i=0}^{n-1} 2^{i-n} \cdot |a_{n-i}|^{1-1} = \sum_{i=1}^{n} 2^{-i} < 1.$$

b) Let $K \subset \mathbb{C}^n$ be compact. The map is obviously continuous and hence $\Phi^{-1}(K)$ closed. K is bounded, i.e. $\exists r$ such that $|z_i| < r$ for all $(z_i)_i \in K$. Let $x = (x_i)_i \in \Phi^{-1}(K)$, then $|\Phi(x_i)| < r$. But x_i is a zero of the polynomial $F(z) = \sum_{k=0}^{n}(-1)^k s_k z^{n-k}$ and hence bounded by part a) by $2\max\{|s_k(x_i)|^{1/k} : 1 \leqslant k \leqslant n\} < \max\{2, 2r\}$. Thus $\Phi^{-1}(K)$ is bounded and therefore compact. Hence Φ is proper.
The map is surjective: Take $a = (a_k)_k \in \mathbb{C}^n, a_0 = 1$ and consider the polynomial $G(z) = \sum_{k=0}^{n}(-1)^k a_k z^{n-k}$. Then $G(z)$ has n complex zeroes $x_1, \ldots, x_n$. Thus $\Phi((x_i)_i) = (a_k)_k$.

Problem 16

Let U be open and $\pi|_{V_i} : V_i \to U$ a local homeomorphism with inverse τ_i. Then $f_i = \tau_i^*(f)$ and $\mathrm{tr}(f) = \sum_i f_i$. We show inductively that all elementary symmetric functions are polynomials of $\mathrm{tr}(f), \mathrm{tr}(f^2)$, a.s.o. We claim that $s_k(f_1, \ldots, f_n) = k^{-1} \sum_{i=1}^{k}(-1)^{i+1} s_{k-i}(f_1, \ldots, f_n)\mathrm{tr}(f^i)$.
The case $k = 1$ is obvious, the case two and three are:

$$s_2 = \frac{1}{2} \cdot (\mathrm{tr}(f)^2 - \mathrm{tr}(f^2)), \quad s_3 = \frac{1}{6} \cdot (\mathrm{tr}(f)^3 - 4 \cdot \mathrm{tr}(f)\mathrm{tr}(f^2) + 2 \cdot \mathrm{tr}(f^3))$$

We have to proof the identity $k s_k(f_1, \ldots, f_n) = \sum_{i=1}^{k}(-1)^{i+1} s_{k-i}(f_1, \ldots, f_n)\mathrm{tr}(f^i)$. Observe that every monomial $\prod_{i=0}^{n} f_i^{e_i}$ on the right hand side with one $r = e_j > 1$ occurs in exactly two summands -

$$(-1)^{r+1} \left(\prod_{i \neq j} f_i^{e_i}\right) \cdot f_j^r \text{ in } (-1)^{r+1} s_{k-r}\mathrm{tr}(f^r), \quad (-1)^r \left(f_j \prod_{i \neq j} f_i^{e_i}\right) \cdot f_j^{r-1} \text{ in } (-1)^r s_{k-r+1}\mathrm{tr}(f^{r-1})$$

with different sign. Hence it trops out. What is left, are those monomials with $e_j = 1$ for exactly k entries. These add up to $k s_k$. (For every ordered tupel $(i_1, \ldots, i_k)$ and every i_j, $\prod_{l \neq j} f_{i_l} \in s_{k-1}$ times one factor f_{i_j} from $\mathrm{tr}(f)$.)