Mathematisches Institut
der Universität München
Pascal Reisert

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Problem sheet 3
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Riemann Surfaces
Solution

Problem 12

a) Choose $\Lambda_0 = \Lambda \cap K_{2r}$ for $K_{2r}, K_r = \{z \in \mathbb{C} : |z| \leqslant r\}$. Then for all $w \in \Lambda \setminus \Lambda_0, \forall z \in K_r$: $|w| > 2|z|$. W.l.o.g. we assume $|w_1| \leqslant |w_2|$: We find a constant c such that

Hence

$$\begin{aligned}\left|\sum_{w\in\Lambda\setminus\Lambda_0} \frac{1}{(z-w)^2} - \frac{1}{w^2}\right| &\leqslant \sum_{w\in\Lambda\setminus\Lambda_0} \frac{|-z^2+2zw|}{|w|^2|z-w|^2} \\ &\leqslant \sum_{w\in\Lambda\setminus\Lambda_0} \frac{|z||2w-z|}{|w|^2||w|-|w|/2|^2} \\ &\leqslant \sum_{w\in\Lambda\setminus\Lambda_0} \frac{4|z|||2w|+|z||}{|w|^4} \\ &\leqslant \sum_{w\in\Lambda\setminus\Lambda_0} \frac{10r}{|w|^3} = \sum_{\substack{nw_1+mw_2\in\Lambda\setminus\Lambda_0\\ n,m\in\mathbb{Z}}} \frac{10r}{|nw_1+mw_2|^3} \\ &\overset{(*)}{\leqslant} \frac{10r}{|w_1||\sin(\angle(w_1,w_2))|} \sum_{\substack{nw_1+mw_2\in\Lambda\setminus\Lambda_0\\ n,m\in\mathbb{Z}}} \frac{1}{||n|+|m||^3} \\ &\leqslant \frac{40r}{|w_1||\sin(\angle(w_1,w_2))|} \sum_{\substack{nw_1+mw_2\in\Lambda\setminus\Lambda_0\\ n,m\in\mathbb{Z}_{\geqslant 0}}} \frac{1}{(n+m)^3} \\ &\overset{(\star)}{\leqslant} \frac{40r}{|w_1||\sin(\angle(w_1,w_2))|} \sum_{\substack{n\in\mathbb{Z}_{\geqslant 0}\\ |n|\geqslant 2r/|w_2|}} \frac{n+1}{n^3} \\ &\leqslant cr \cdot \int_{x\geqslant 2r/|w_2|} \frac{x+1}{x^3}\,\mathrm{d}\,x \\ &\leqslant cr \cdot \left(\frac{|w_2|}{2r} + \frac{|w_2|^2}{8r^2}\right) \\ &< \infty.\end{aligned}$$

Thus the term converges uniformely on every K_r. Hence $\wp := \wp_\Lambda$ is a meromorphic function with poles of order two in the lattice points.

Remark. $(*)$ results from rotating the lattice such that $w_1 \in \mathbb{R}$ and then taking the norm. For $(\star)$ one counts the number of (standard) lattice points with 1-norm n. There are $n+1$ of these points. Since the 1-norm is smaller than the euclidean norm (in this case) we may restrict to $|n| \geqslant 2r/|w_2|$. Finally we write our sum as an integral over a step function $\sum_{n\geqslant\lceil 2r/|w_2|\rceil} \frac{n+1}{n^3} \cdot 1_{[n,n+1[}$ and set $c = \frac{8\cdot 40r}{|w_1||\sin(\angle(w_1,w_2))|}$. The factor 8 ensures that we get for the integrants $8 \cdot \frac{x+1}{x^3} \geqslant \frac{n+1}{n^3} \cdot 1_{[n,n+1[}(x), x \geqslant 0$. Observe that $\left(\frac{1+\lceil 2r/|w_2|\rceil}{\lceil 2r/|w_2|\rceil}\right)^3 \leqslant 8$ as $\lceil 2r/|w_2|\rceil \geqslant 1$.