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Problem sheet 2
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Riemann Surfaces
Solution

Problem 6

a) Note that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}, \partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. We have $\overline{\partial}\varphi = 0$. The chain rule for differentiable functions f, g, i.e. $\partial(f \circ g) = (\partial f)(g) \cdot \partial g + (\overline{\partial} f)(g) \cdot \partial \overline{g}$, can be read of the following calculation

$$\begin{aligned} d(f \circ g) &= (\partial f)(g) \cdot dg + (\overline{\partial} f)(g) \cdot d\overline{g} \\ &= (\partial f)(g) \cdot (\partial g dz + \overline{\partial} g d\overline{z}) + (\overline{\partial} f)(g) \cdot (\partial \overline{g} dz + \overline{\partial}\,\overline{g} d\overline{z}) \\ &= \left((\partial f)(g) \cdot \partial g + (\overline{\partial} f)(g) \cdot \partial \overline{g}\right) dz + \left((\partial f)(g) \cdot \overline{\partial} g + (\overline{\partial} f)(g) \cdot \overline{\partial}\,\overline{g}\right) d\overline{z} \\ &= \partial(f \circ g) dz + \overline{\partial}(f \circ g) d\overline{z}. \end{aligned}$$

Using the chain rule we get

$$\begin{aligned} \partial\overline{\partial}(f \circ \varphi) &= \partial((\partial f)(\varphi) \cdot \underbrace{\overline{\partial}\varphi}_{=0} + (\overline{\partial} f)(\varphi) \cdot \overline{(\partial \varphi)}) \\ &= (\underbrace{\partial\overline{\partial} f}_{=0})(\varphi) |(\partial\varphi)|^2 + (\overline{\partial}^2 f)(\varphi) \cdot (\underbrace{\partial \overline{\varphi}}_{=0}) \cdot \overline{(\partial\varphi)} + (\overline{\partial} f)(\varphi) \cdot \underbrace{\partial\overline{(\partial\varphi)}}_{=0}) = 0. \end{aligned}$$

We used that φ and $\partial\varphi$ are holomorphic - $\overline{\partial}\varphi = 0$ resp. $\overline{\partial}\partial\varphi = 0$ - and that f is harmonic, i.e. $\partial\overline{\partial} f = 0$.

Alternatively we will present a proof using the two-dimensional real chain rule and the Cauchy-Riemann differential equations. To this end, let $\varphi = \psi + i\tau$ with real-valued functions ψ, τ. We calculate

$$\begin{aligned} &\frac{\partial^2 f \circ \varphi}{\partial x^2} \\ &= \frac{\partial^2 u(\psi + i\tau) + iv(\psi + i\tau)}{\partial x^2} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \psi}(\psi + i\tau) \cdot \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial \tau}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial x}\right) \\ &= \frac{\partial^2 f}{\partial^2 \psi}(\psi + i\tau) \cdot \left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{\partial f}{\partial \tau \partial \psi}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial x} \cdot \frac{\partial \psi}{\partial x} \\ &\quad + \frac{\partial^2 f}{\partial \tau^2}(\psi + i\tau) \cdot \left(\frac{\partial \tau}{\partial x}\right)^2 + \frac{\partial f}{\partial \psi \partial \tau}(\psi + i\tau) \cdot \frac{\partial \psi}{\partial x} \cdot \frac{\partial \tau}{\partial x} \\ &= \frac{\partial^2 f}{\partial^2 \psi}(\psi + i\tau) \cdot \left(\frac{\partial \psi}{\partial x}\right)^2 + 2 \cdot \frac{\partial f}{\partial \tau \partial \psi}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial x} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial^2 f}{\partial \tau^2}(\psi + i\tau) \cdot \left(\frac{\partial \tau}{\partial x}\right)^2 \end{aligned}$$

$$
\begin{aligned}
&\frac{\partial^2 f \circ \varphi}{\partial y^2} \\
&= \frac{\partial^2 u(\psi + i\tau) + iv(\psi + i\tau)}{\partial y^2} \\
&= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial \psi}(\psi + i\tau) \cdot \frac{\partial \psi}{\partial y} + \frac{\partial f}{\partial \tau}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial y} \right) \\
&= \frac{\partial^2 f}{\partial^2 \psi}(\psi + i\tau) \cdot \left(\frac{\partial \psi}{\partial y} \right)^2 + 2 \cdot \frac{\partial f}{\partial \tau \partial \psi}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial y} \cdot \frac{\partial \psi}{\partial y} + \frac{\partial^2 f}{\partial \tau^2}(\psi + i\tau) \cdot \left(\frac{\partial \tau}{\partial y} \right)^2
\end{aligned}
$$

Putting both equations together we receive using the Cauchy-Riemann PDEs, i.e. $\partial\psi/\partial x = \partial\tau/\partial y, \partial\psi/\partial y = -\partial\tau/\partial x$

$$
\begin{aligned}
&\frac{\partial^2 f \circ \varphi}{\partial x^2} + \frac{\partial^2 f \circ \varphi}{\partial y^2} \\
&= \frac{\partial^2 f}{\partial^2 \psi}(\psi + i\tau) \cdot \left(\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right) + 2 \cdot \frac{\partial f}{\partial \tau \partial \psi}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial x} \cdot \frac{\partial \psi}{\partial x} \\
&\quad + \frac{\partial^2 f}{\partial \tau^2}(\psi + i\tau) \cdot \left(\left(\frac{\partial \tau}{\partial x} \right)^2 + \left(\frac{\partial \tau}{\partial y} \right)^2 \right) - 2 \cdot \frac{\partial f}{\partial \tau \partial \psi}(\psi + i\tau) \cdot \frac{\partial \tau}{\partial x} \cdot \frac{\partial \psi}{\partial x} \\
&= \left(\frac{\partial^2 f}{\partial^2 \psi} + \frac{\partial^2 f}{\partial^2 \tau} \right) (\psi + i\tau) \cdot \left(\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right) \\
&= 0
\end{aligned}
$$

for a harmonic function f.