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Problem sheet 12
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Riemann Surfaces
Solution

Problem 45

Analogous to the discussion of problem 19 we see that X has exactly n branch points $(e^{2\pi ij/n},0), 0 \leqslant j < n$ of order n. Hence the ramification index is $b = n(n-1)$. By Riemann-Hurwitz we get $g = \frac{n(n-1)}{2} + n(-1) + 1 = \frac{n^2-3n+2}{2} = \frac{(n-1)(n-2)}{2}$.

Problem 46

We get four branch points $(a_j, 0), 1 \leqslant j \leqslant 4$. X has a branch point at ∞: For $u = z^{-1}$ we get $w = \sqrt[3]{u^{-1}}u^{-1}\sqrt[3]{\prod_{j=1}^4(1-ua_j)} = \sqrt[3]{u^{-1}}\frac{h(u)}{u}$ for a holomorphic non-vanishing function h. Thus we get a branch point at ∞. Hence $b = 5(3-1) = 10$ and $g = 10/2 + 3(-1) + 1 = 3$. Furthermore note that $\dim H^0(X,\Omega) = \dim H^1(X,\mathcal{O}) = \dim H^0(X,\mathcal{O}) - 1 + g = g = 3$. The differential forms are obviously not linearly independent, thus it is enough to show that they are holomorphic on X. The σ_i are holomorphic outside the branch points. Now choose a local coordinate u_j with $u_j^3 = z - a_j$. Then

$$\sigma_1 = \frac{dz}{w} = \frac{3u_j^2 du_j}{u_j h_j(u_j)} = \frac{3u_j du_j}{h_{1j}(u_j)}, \quad \sigma_2 = \frac{3u_j^2 du_j}{u_j^2 h_j^2(u_j)} = \frac{3du_j}{h_{1j}(u_j)}, \quad \sigma_3 = \frac{(u_j^3 + a_j)du_j}{h_j^2(u_j)}$$

for a holomorphic non-vanishing function h_j. Finally we have to check holomorphicity at ∞. Take a coordinate u_∞ with $u_\infty^3 = z^{-1}$. We get

$$\sigma_1 = \frac{dz}{w} = \frac{-3u_\infty^{-4}du_\infty}{\sqrt[3]{u_\infty^{-3}}\frac{h(u_\infty^3)}{u_\infty^3}} = -\frac{3du_\infty}{h(u_\infty^3)}, \quad \sigma_2 = -\frac{3u_\infty^4 du_\infty}{h(u_\infty^3)}, \quad \sigma_3 = -\frac{3u_\infty du_\infty}{h(u_\infty^3)}.$$

Problem 47

a) The long exact sequence to the given short exact sequence is

$$\mathcal{M}(X) \to \mathcal{Q}(X) \to H^1(X,\mathbb{C}) \to H^1(X,\mathcal{M}) = 0.$$

and we get $H^1(X,\mathbb{C}) \simeq \mathcal{Q}(X)/d\mathcal{M}(X)$.

b) By the theorem of deRham-Hodge we get $H^1(X,\mathbb{C}) \simeq \mathrm{Harm}^1(X) = \Omega(X) \oplus \overline{\Omega}(X)$ and $\dim(\mathrm{Harm}^1(X)) = 2g = 2$. Since $dz \notin \mathbb{C}\cdot\wp_\Lambda dz$ we get a basis $(dz, \wp_\Lambda dz)$.

Problem 48

a) The divisor D has finite support on X and hence on U. On U we find a meromorphic function f_D with divisor $D|_U$, e.g. $\prod_{x\in\mathrm{Supp}(D)}(z-x)^{D(x)}$. Then $\mathcal{O}_D \xrightarrow{\simeq} \mathcal{O}, g \mapsto gf_D$ is an isomorphism. Hence $H^1(U,\mathcal{O}_D) \simeq H^1(U,\mathcal{O})$. Construct a cover of $X \setminus \{a\}$ as follows: Take a punctured disc neighbourhood V_0 around a. Let $V_i, i \in \mathbb{Z}_+$ be small discs such that $a \notin \overline{V_i \cap V_0}$. As we have seen before $H^1(V_i,\mathcal{O}_D) = 0$ for discs and $H^1(V_0,\mathcal{O}_D) = 0$ for a ring area. Thus $\mathfrak{U} = (U_i)_{i\in\mathbb{N}}$ is a Leray cover of $X \setminus \{a\}$. Now take a cocycle $(f_{ij}) \in Z^1(\mathfrak{U},\mathcal{O}_D)$. We add another chart V_0' such that $a \in V_0' \subset V_0, D|_{V_0'} = D(a)$ and $V_0' \cap V_i = \emptyset$ for $i > 0$. We get a Leray covering $\mathfrak{U}'$ of X. Since there are no triple intersections with V_0' we can extend (f_{ij}) to a cocycle $(f'_{ij}) \in Z^1(\mathfrak{U}',\mathcal{O}_D)$ by choosing $f_{0'0}$ aribitrary holomorphic on $V_0'\setminus\{a\}$. But then $(f'_{ij}) \in Z^1(\mathfrak{U}',\mathcal{O}_{D+ka})$ for every $k \in \mathbb{Z}$. If k is large enough such that $\deg(D+ka) > 2g-2$, then $H^1(\mathfrak{U}',\mathcal{O}_{D+ka}) = 0$. Thus $f_{ij} = f_i - f_j$ for $(f_i) \in C^0(\mathfrak{U}',\mathcal{O}_{D+ka})$. The reduced chain $(f_i) \in C^0(\mathfrak{U},\mathcal{O}_{D+ka}) = C^0(\mathfrak{U},\mathcal{O}_D)$ satisfies $f_{ij} = f_i - f_j$ and hence $H^1(X \setminus \{a\}) = 0$.

b) *Der folgende Beweis stammt von Ludwig Fürst:* We take the Leray cover $\mathfrak{U}$ of part a). A cocylce looks like $f = \sum_{k\in\mathbb{Z}} c_k z^k$ on the punctured neighbourhood $U \setminus \{a\}$. Since $H^1(\mathfrak{U},\mathcal{O}_{(2g-1)a}) = H^1(X,\mathcal{O}_{(2g-1)a}) = 0$ we know that $f = g - h$ with $g \in \mathcal{O}_{(2g-1)a}(U)$ and $h \in \mathcal{O}_{(2g-1)a}(V) = \mathcal{O}(V)$. Furthermore we find a holomorphic function $g' \in \mathcal{O}(U)$ such that $f + h - g' = \sum_{k=1}^{2g-1} b_k z^{-k} \in \mathcal{O}_{(2g-1)a}$ the principal part of $f + h$. We immediately get $H^1(X,\mathcal{O}) = \mathbb{C}^{2g-1}/L$.