Mathematisches Institut
der Universität München
Pascal Reisert

Summer Term 2016
Problem sheet 10
June 22nd, 2016

Riemann Surfaces
Solution

Problem 40

c) We reproduce a proof by Ludwig Fürst with a slightly different notation:
Choose local coordinates z and $w = z^k$. Obvserve that $dw = kz^{k-1}dz$ and $\varphi_j(w) = e^{2\pi ij/k}w^{1/k}$ w.l.o.g.. Let $\omega = \sum_{l\in\mathbb{Z}} c_l z^l\, dz$. Then

$$\mathrm{Tr}(\omega) = \sum_{l\in\mathbb{Z}}\sum_{j=1}^{k} c_l \left(e^{2\pi ij/k}w^{1/k}\right)^l \frac{dw}{k\left(e^{2\pi ij/k}w^{1/k}\right)^{k-1}} = \sum_{l\in\mathbb{Z}}\sum_{j=1}^{k} \frac{c_l}{k}\cdot e^{2\pi ij(l-k+1)/k}w^{(l-k+1)/k}\, dw.$$

If $\frac{l-k+1}{k} \in \mathbb{Z} \Leftrightarrow \frac{l+1}{k} \in \mathbb{Z}$, then $e^{2\pi ij(l-k+1)/k} = 1$ for all j and we get

$$\sum_{j=1}^{k} \frac{c_l}{k}\cdot e^{2\pi ij(l-k+1)/k}w^{(l-k+1)/k} = \sum_{j=1}^{k} \frac{c_l}{k}\cdot w^{(l-k+1)/k} = c_l \cdot w^{(l-k+1)/k}.$$

If $\frac{l-k+1}{k} \notin \mathbb{Z}$, then by the geometric sum formula $\sum_{j=1}^{k} e^{2\pi ij(l-k+1)/k} = \frac{e^{2\pi ij(l-k+1)k/k}-1}{e^{2\pi i(l-k+1)/k}-1} = 0$. Hence we get

$$\sum_{j=1}^{k} \frac{c_l}{k}\cdot e^{2\pi ij(l-k+1)/k}w^{(l-k+1)/k} = 0.$$

Putting both results together we get for $l-k+1 = mk \Leftrightarrow l = (m+1)k-1$

$$\mathrm{Tr}(\omega) = \sum_{m\in\mathbb{Z}} c_{(m+1)k-1}w^m\, dw.$$

a) Follows from part c) since $m < 0 \Rightarrow (m+1)k-1 < 0$.

b) Follows from part c) for the $m = -1$ coefficient $c_{(-1+1)k-1} = c_{-1}$.