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Mathematical Gauge Theory I

Sheet 12

Exercise 1. Let V be an oriented 4-dimensional $\mathbb{R}$-vector space with a scalar product.

a) Given an oriented orthonormal basis $\alpha_1, ..., \alpha_4 \in V$, write out explicit orthonormal bases for $\Lambda^2_+ V$ and $\Lambda^2_- V$, derived from the α_i.

b) Given an arbitrary vector space W and an $\alpha \neq 0 \in V$, show the linear map

$$\pi_- \circ \alpha \otimes : V \otimes W \longrightarrow (\Lambda_- V) \otimes W$$
$$\beta \otimes w \longmapsto (\alpha \wedge \beta)_- \otimes w$$

has kernel consisting of elements of the form $\alpha \otimes w \in V \otimes W$.

c) Show that the map in *b)* is surjective

Remark: This completes the proof that the symbol sequence of the twisted half de Rham complex of an oriented Riemannian 4-manifold is exact.

Exercise 2. Let $Q : \mathbb{Z}^r \times \mathbb{Z}^r \longrightarrow \mathbb{Z}$ be a positive definite symmetric bilinear form. Assume Q is unimodular in the sense that $\det Q = \pm 1$. Let m be one half the number of solutions to the equation

$$Q(\alpha, \alpha) = 1.$$

a) Prove that $m \leq r$, with equality if, and only if, Q can be diagonalized over $\mathbb{Z}$.

b) Prove that the symmetric bilinear form corresponding to the quadratic form

$$Q_{E_8}(x_1, ..., x_8) := 2 \sum_{i=1,...,8} x_i^2 - 2 \sum_{i=1,...,6} x_i x_{i+1} - 2x_5 x_8$$

is positive definite and unimodular, but not diagonalizable over $\mathbb{Z}$.

Exercise 3. For a closed oriented 4-manifold X, denote by Q_X its intersection form.

a) Compute $Q_{S^2 \times S^2}$ and $Q_{\mathbb{CP}^2}$.

b) Let $P(x, y) := x^2 - y^2$. Show that $Q_{S^2 \times S^2}$ is equivalent to P over the reals, but not over the integers.

Remark for those who know about connected sums: One may see check $P = Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}$. So even though $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ have the same Betti-numbers, they are distinguished by their intersection form.

Hand in: during the exercise classes.