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Mathematical Gauge Theory I

Sheet 11

Exercise 1. Let ∇ be a covariant derivative on $E \to B$ and $\overline{\nabla}$ its extension on End$E$ defined by

$$(\overline{\nabla}_X \varphi)s = \nabla_X(\varphi(s)) - \varphi(\nabla_X s)$$

for all $X \in TB, s \in \Gamma(E), \varphi \in \Gamma(\mathrm{End}E)$.

a) Prove that $\overline{\nabla}$ is indeed a covariant derivative on End$E$.

b) Prove that

$$F^{\overline{\nabla}}(X,Y)\varphi = [F^{\nabla}(X,Y), \varphi]$$

where the right-hand side is the commutator of endomorphism

$$[\psi, \varphi] = \psi \circ \varphi - \varphi \circ \psi.$$

Exercise 2. Let $P \to B$ be a principal G-bundle.

a) Prove that if P admits a reduction to $S^1 \subset G$, then P admits a Yang-Mills connection for any Riemannian metric on B.

b) If B is 4-dimensional, is the same statement true for self-dual or anti-self-dual Yang-Mills connections?

Exercise 3. Let $P \to B$ be a principal G-bundle with gauge group $\mathcal{G}$ and space of connections $\mathcal{C}$. Determine all possible stabilizers $\mathrm{Stab}(\omega) \subset \mathcal{G}$ for the $\mathcal{G}$-action on $\mathcal{C}$ in the cases $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$.

Exercise 4. Consider the principal SU(2)-bundle $S^7 \to S^4$ defined in an analogous way to the Hopf bundle $S^3 \to S^2$ when replacing the complex number with quaternions.

a) In analogy with Exercise 4 in Sheet 3 define a connection 1-form $A \in \Omega^1(S^7, \mathfrak{su}(2))$.

b) Prove that A satisfies the Yang-Mills equation for the standard round Riemannian metric on S^4.

Hand in: during the exercise classes.