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Mathematical Gauge Theory I

Sheet 10

Exercise 1. Let (M, g) be an n-dimensional oriented Riemannian manifold and $*$ the Hodge star operator.

1. Prove that
$$** : \Omega^k(M) \longrightarrow \Omega^k(M)$$
is given by
$$** = (-1)^{k(n-k)}$$

2. Determine the even dimensions $n = 2k$ where $** = 1$ on $\Omega^k(M)$. In these dimensions we can define self-dual and anti-self-dual k-forms ω, satisfying $*\omega = \omega$ and $*\omega = -\omega$, respectively.

Exercise 2. Let (M, g) be a closed (compact without boundary) n-dimensional oriented Riemannian manifold. The Laplace operator on k-forms is defined by
$$\Delta = dd^* + d^*d : \Omega^k(M) \longrightarrow \Omega^k(M)$$
where d^* is the formal adjoint of d. A form ω is called harmonic if $\Delta\omega = 0$. Prove that
$$\omega \text{ is harmonic} \iff d\omega = 0 = d^*\omega \iff *\omega \text{ is harmonic.}$$

Exercise 3. Let (M, g) be a Riemannian 4-manifold with principal bundle $P \longrightarrow M$. Prove that the Yang-Mills functional is invariant under conformal change of the metric, i.e. when replacing g by g' with
$$g' = e^{2\lambda} g,$$
where $\lambda \in C^\infty(M)$ is an arbitrary smooth function on M.

Exercise 4.

1. Prove that the connection A on the Hopf bundle $S^3 \longrightarrow S^2$, introduced in Ex. 4, sheet 3, satisfies the Yang-Mills equation if S^2 has the standard round Riemannian metric.

2. Prove that the Yang-Mills moduli space for the Hopf bundle $S^3 \longrightarrow S^2$ over the round sphere S^2 consists of a single point.

Hand in: during the exercise classes.