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Mathematical Gauge Theory I

Sheet 9

Exercise 1. Consider $\mathbb{R}^3$ with the standard scalar product $\langle\ .\ ,\ ,\ \rangle_0$ and compatible flat covariant derivative ∇^0 on $T\mathbb{R}^3$.
For the unit sphere $S^2 \subset \mathbb{R}^3$, we consider

$$TS^2 \subset T\mathbb{R}^3|_{S^2} = TS^2 \oplus \mathbb{R}$$

with the trivial summand spanned by the outward unit normal N to $S^2 \subset \mathbb{R}^3$. On $TS^2 \to S^2$, we define a covariant derivative ∇ by

$$\nabla_X Y = \mathrm{pr}(\nabla^0_X Y)$$

where $\mathrm{pr} : T\mathbb{R}^3|_{S^2} \longrightarrow TS^2$ is the projection with kernel spanned by N.

a) Check that ∇ is compatible with the metric $\langle\ .\ ,\ .\ \rangle$ on TS^2 given by the restriction of $\langle\ .\ ,\ .\ \rangle_0$.

b) Compute an explicit representative for $e(TS^2) \in H^2_{dR}(S^2)$ from the curvature of ∇.

c) Prove that $\int_{S^2} e(TS^2) = 2$.

Hint: $f : [0, 2\pi] \times [0, \pi] \to S^2, (u, v) \mapsto (\cos u \cdot \sin v, \sin u \cdot \sin v, \cos v)$ gives a parametrization of S^2. Normalize $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$ to unit length and check that this gives an orthonormal frame for TS^2. The calculation in *b)* and *c)* is easy using this frame.

Exercise 2. Let $P \longrightarrow B$ be a principal $SO(2)$-bundle. Give a definition of an Euler class $e(P) \in H^2(B, \mathbb{R})$ which does not use the associated vector bundle, but instead a connection 1-form and local expression for the curvature of this form. Prove that it is independent of the choices made. Then show that the class you defined coincides with the Euler class of the associated vector bundle. Use this and Exercise 1 from sheet 4 to compute $\int_{\mathbb{CP}^1} e$, where e is the Euler class of the Hopf bundle.

Exercise 3. Show that the Euler class is functorial under pullbacks, i.e. given a smooth map $f : N \longrightarrow M$ and an oriented rank 2 bundle E over M, one has:

$$e(f^*E) = f^*e(E).$$

(please turn)

Exercise 4. Let B be a manifold and let E be an oriented vector bundle with a decomposition $E = L \oplus \underline{\mathbb{R}}$, where L is a line bundle and $\underline{\mathbb{R}}$ the trivial line bundle. Show that $e(E) = 0$. Deduce that for an arbitrary E (still oriented, of rank two) the Euler class $e(E)$ vanishes if there exists a nowhere vanishing section $s : B \longrightarrow E$.

Hand in: during the exercise classes.