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Mathematical Gauge Theory I

Sheet 8

Exercise 1. Let P be a manifold and $\mathfrak{g}$ a Lie algebra. Recall that for $\omega \in \Omega^k(P, \mathfrak{g})$ and $\eta \in \Omega^l(P, \mathfrak{g})$ we defined $[\omega, \eta] \in \Omega^{k+l}(P, \mathfrak{g})$ by

$$[\omega, \eta](X_1, ..., X_l) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}[\omega(X_{\sigma(1)}, ..., X_{\sigma(k)}, \eta(X_{\sigma(k+1)}, ..., X_{\sigma(n)}].$$

Prove the that this pairing has the following properties:

a) $[\omega, \eta] = -(-1)^{kl}[\eta, \omega]$

b) $[\eta, [\eta, \eta]] = 0$

c) $d[\omega, \eta] = [d\omega, \eta] + (-1)^k[\omega, d\eta]$

Exercise 2. Let P be a principal G-bundle and $\mathfrak{g}$ be the Lie algebra of G. If $\omega \in \Omega^1(P, \mathfrak{g})$ is a connection 1-form with curvature Ω, prove the Bianchi-identity

$$d\Omega = [\Omega, \omega].$$

Deduce from this the form of the Bianchi identity proved in the lectures:

$$d\Omega|_{\ker \omega} \equiv 0.$$

Exercise 3. In the setting of the previous exercise, Ω corresponds to some $F \in \Omega^2(B, \operatorname{Ad}(P))$, where $\operatorname{Ad}(P) = P \times_{\operatorname{Ad}} \mathfrak{g}$. The form ω induces a covariant derivative ∇ on $\Gamma(\operatorname{Ad}(P))$, which is extended to $\overline{\nabla}$ on $\Omega^k(B, \operatorname{Ad}(P))$.
Prove

$$\overline{\nabla} F = 0.$$

Exercise 4. We will continue exercise 4 from the previous sheet:

d) Let $\rho_1, \rho_2 \colon \pi_1(B) \to G$ be two homomorphism and define $P_i = P_{\rho_i}$ and $\omega_i = \omega_{\rho_i}$ as before. Prove that there exists an isomorphism $\phi \colon P_1 \to P_2$ with $\phi^*\omega_2 = \omega_1$ if and only if there exists $g \in G$ such that $\rho_2 = g\rho_1 g^{-1}$.

Hand in: during the exercise classes.