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Mathematical Gauge Theory I

Sheet 7

Exercise 1. Let E be a vector bundle with covariant derivative ∇. For two local trivializations differing by a gauge tranformation g prove that the two curvature matrices are related by $\Omega' = g\,\Omega\,g^{-1}$.

Exercise 2. Let E be a vector bundle with covariant derivative ∇ and F^{∇} its curvature. Prove that

$$F^{\nabla}(X,Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

[Hint: You may use the special case proved during the lecture.]

Exercise 3. Let $P \to B$ be a principal G-bundle and $\varphi\colon G \to H$ a homomorphism between Lie groups. Denote by P_φ the associated principal H-bundle. Show that for every connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ there exists a unique connection 1-form $\omega' \in \Omega^1(P_\varphi, \mathfrak{h})$ such that

$$f^* \omega' = \varphi_* \circ \omega$$

where $f\colon P \to P_\varphi$ is defined by $f(p) = [(p, e)]$.
[Hint: Use Exercise 1 from Sheet 3.]

(please turn)

Exercise 4. Let G and $\pi\colon \widetilde{B} \to B$ be as in Sheet 5, Exercise 2. Denote by $q\colon P_\rho \to B$ the principal G-bundle associated to the universal covering by $\rho\colon \pi_1(B) \to G$. Define the map $f\colon \widetilde{B} \to P_\rho$ by $f(p) = [(p, e)]$ as in Exercise 3.

a) Use the previous exercise to show that there is a unique flat connection 1-form $\omega_\rho \in \Omega^1(P_\rho, \mathfrak{g})$ such that $f^*\omega_\rho = 0$.

b) Let $q_P\colon P \to B$ be a principal G-bundle equipped with a connection 1-form ω_P and $f_1, f_2\colon \widetilde{B} \to P$ two maps such that $q_P \circ f_i = \pi$ and $f_i^*\omega_P = 0$. Show that there exists a unique $g \in G$ such that $f_2 = f_1 g$.

c) Let $q_P\colon P \to B$ be a principal G-bundle equipped with a connection 1-form ω_P and $f\colon \widetilde{B} \to P$ a map satisfying $q_P \circ f = \pi$ and $f^*\omega_P = 0$. Show that there exists a homomorphism $\rho_f\colon \pi_1(B) \to G$ with $f \circ \gamma = f\rho_f(\gamma)^{-1}$ for all $\gamma \in \pi_1(B)$ such that the map

$$\begin{aligned} \widetilde{B} \times G &\longrightarrow P \\ (p, g) &\mapsto f(p)g \end{aligned}$$

induces an isomorphism $\phi\colon P_{\rho_f} \to P$ with $\phi^*\omega_P = \omega_{\rho_f}$.

Hand in: during the exercise classes.