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Mathematical Gauge Theory I

Sheet 6

Exercise 1. Let $\pi\colon P \to B$ be a principal G-bundle and $E = P \times_\rho V$ an associated vector bundle. Fix a connection ω on P and let ∇ be the induced connection on E. Using the definition of ∇ show that

$$\nabla_X(fs) = f\nabla_X s + (\mathcal{L}_X f)s$$

for any $f \in C^\infty(B)$, $s \in \Gamma(E)$ and $X \in \mathfrak{X}(B)$.

Exercise 2. Let $\pi\colon P \to B$ be a principal G-bundle where the Lie group G is abelian. Denote by $C^\infty(B, G)$ the group of smooth maps from B to G with multiplication given by pointwise multiplication and by $C^\infty(P, G)^G$ be the group $\{f\colon P \to G | \, f(pg) = g^{-1}f(p)g\}$. Show that the following map is a group isomorphism

$$\begin{aligned} C^\infty(B, G) &\to C^\infty(P, G)^G \\ \sigma &\mapsto f_\sigma = \sigma \circ \pi. \end{aligned}$$

Exercise 3. Let $\pi\colon P \to B$ be a principal G-bundle and $E = P \times_G F$ be an associated fibre bundle. Recall that a map $f\colon M \to B$ defines a pullback bundle $f^*P \to M$, cf. Exercise 1 from Sheet 2.

a) Show that the pullback principal G-bundle $\pi^*P \to P$ is trivial.

b) Show that E is trivial if P is trivial.

c) Show that the pullback bundle $\pi^*E \to P$ is always trivial.

Exercise 4. Let $\pi\colon P \to B$ be a principal G-bundle and $\omega \in \Omega^1(P, \mathfrak{g})$ a connection 1-form on P. Suppose that $\sigma \in \mathcal{G}$ is a bundle automorphism, i.e. a gauge transformation. Prove that $\sigma^*\omega$ is a connection 1-form on P which satisfies

$$\sigma^*\omega = \mathrm{Ad}(f^{-1})\omega + f^*\theta$$

where f is the unique function such that $\sigma(p) = pf(p)$ and θ is the tautological 1-form on G.

Hand in: during the exercise classes.