Fall term 2019

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Mathematical Gauge Theory I

Sheet 5

Exercise 1.

Let G be a Lie group and $Ad\colon G \to GL(\mathfrak{g})$ its adjoint representation. Show that the differential of Ad at the identity element is given by

$$\begin{aligned} D_e Ad\colon \mathfrak{g} &\longrightarrow \quad \mathrm{End}(\mathfrak{g}) \\ X &\mapsto (Y \mapsto [X,Y]) \end{aligned}$$

Exercise 2. Let $\widetilde{B} \to B$ be the universal covering space of a smooth connected manifold B. Regard $\widetilde{B}$ as a principal bundle with fibre the discrete Lie group $\pi_1(B)$ acting on the right. Let G be a Lie group and consider the bundle $\widetilde{B} \times_{\pi_1(B)} G$ associated to $\widetilde{B}$ by $\rho\colon \pi_1(B) \to G$ with $\pi_1(B)$ acting on G by ρ composed with left multiplication..

a) Show that $\widetilde{B} \times_{\pi_1(B)} G$ is a principal G-bundle over B which admits a flat connection.

b) Show that for any representation $\rho\colon \pi_1(B) \to GL(n, \mathbb{R})$ the associated vector bundle $\widetilde{B} \times_\rho \mathbb{R}^n$ admits an integrable connection, i.e. an integrable horizontal subbundle $H \subset T(\widetilde{B} \times_\rho \mathbb{R}^n)$.

Exercise 3. Let $\pi : E \to B$ be a rank k vector bundle with fiber V over a smooth manifold B.

a) Prove that the set of bases, or frames, in the fibres of E naturally forms a principal $GL(V)$-bundle over B (the **frame bundle** $Fr(E)$).

b) Show that the vector bundle $Fr(E) \times_\rho V$ associated to $Fr(E)$ by the tautological representation $\rho\colon GL(k, \mathbb{R}) \to GL(V)$ is isomorphic to E.

Exercise 4. Let $\pi\colon P \to B$ be a principal G-bundle.

a) Consider two representations $\rho, \rho'\colon G \to GL(n, \mathbb{R})$. Show that the associated vector bundles $P \times_\rho \mathbb{R}^n$ and $P \times'_\rho \mathbb{R}^n$ are isomorphic if and only if there exists a smooth map $\phi\colon P \to GL(n, \mathbb{R})$ satisfying $\rho'_g \phi_{pg} = \phi_p \rho_g$.

b) Analogously prove that, for two smooth actions $\mu\colon G \times F \to F$ and $\mu'\colon G \times F \to F$, the associated fibre bundles $P \times_{G,\mu} F$ and $P \times_{G,\mu'} F$ are isomorphic if and only if there exists $\phi\colon P \to \mathrm{Diff}(F)$ satisfying $\mu'_g \phi_{pg} = \phi_p \mu_g$.

Hand in: during the exercise classes.