Fall term 2019

Prof. D. Kotschick
Dr. J. Stelzig
G. Placini

Mathematical Gauge Theory I

Sheet 4

Exercise 1. This is a continuation of Exercise 4 on sheet 3. We will use the same notation.

1. Show that the curvature of the connection 1-form A on the Hopf bundle is given by
$$\Omega^A = -(\alpha_0 \wedge \bar{\alpha}_0 + \alpha_1 \wedge \bar{\alpha}_1)$$

2. Define a 2-form on $\mathbb{C}$ by
$$\tilde{\Omega}_w := -\frac{1}{(1+|w|^2)^2} dw \wedge d\bar{w}.$$
Let $U_1 := \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\}$ and $\psi_1 : U_1 \longrightarrow \mathbb{C}$ be given by $[z_0 : z_1] \longmapsto z_0/z_1$. Show that $\psi_1^*\tilde{\Omega}$ can be prolonged to a form $\Omega_{\mathbb{CP}^1}$ on all of $\mathbb{CP}^1$ which satisfies $\pi^*\Omega_{\mathbb{CP}^1} = \Omega^A$. Deduce that for every local section s of π, defined on some open set $V \subset \mathbb{CP}^1$ one has $s^*\Omega_A = \Omega_{\mathbb{CP}^1}|_V$.

3. Compute the integral $\int_{\mathbb{CP}^1} \Omega_{\mathbb{CP}^1}$.

Exercise 2. Let G be a Lie group and $H \subset G$ a closed subgroup, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. We know that $\pi : G \to G/H$ is an H-principal bundle. Assume that there exists a vectorspace complement $\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g}$ such that $Ad(H)\mathfrak{m} \subset \mathfrak{m}$.

1. Consider $\omega = \pi_{\mathfrak{h}} \circ \theta \in \Omega^1(G, \mathfrak{h})$, where θ is the tautological 1-form on G with values in $\mathfrak{g}$. Prove that ω is a connection 1-form on $G \to G/H$.

2. Show that the vertical and horizontal subspaces defined by ω at a point $g \in G$ are given by $DL_g\mathfrak{h}$ and $DL_g\mathfrak{m}$.

3. Prove that the curvature of the connection ω is given by
$$\Omega = -\frac{1}{2}\pi_{\mathfrak{h}} \circ [\pi_{\mathfrak{m}} \circ \theta, \pi_{\mathfrak{m}} \circ \theta] \in \Omega^2(G, \mathfrak{h}),$$
where the commutator is taken in $\mathfrak{g}$.

Exercise 3. Let $P \to B$ be a principal G-bundle. Assume P admits a reduction of the structure group to a closed Lie subgroup $K \subset G$. Prove that P admits a connection with holonomy contained in K.

Hand in: during the exercise classes.