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Mathematical Gauge Theory I

Sheet 3

Exercise 1. Let $\pi\colon P \to B$ be a principal G-bundle, $\{U_i\}$ an open cover of B by trivializing sets and ω_i 1-forms on U_i with values in $\mathfrak{g}$. Show that if

$$\omega_j = \mathrm{Ad}(\psi_{ij}^{-1})\omega_i + \theta_{ij} \text{ on } U_i \cap U_j$$

then there is a unique connection 1-form ω on P such that $\omega_i = s_i^*\omega$.
[Remark: Here θ_{ij}, ψ_{ij} and s_i are defined as in the lectures.]

Exercise 2. Let $\pi\colon P \to B$ be a principal G-bundle and fix a connection $H \subset TP$. Consider vector fields $V, W \in \mathfrak{X}(B)$ and let $\widetilde{V}, \widetilde{W}$ be their horizontal lifts.

a) Show that $\widetilde{V + W} = \widetilde{V} + \widetilde{W}$.

b) Show that $\widetilde{fV} = (f \circ \pi)\widetilde{V}$ for $f \in C^\infty(B)$.

c) Show that $\widetilde{[V, W]} = [\widetilde{V}, \widetilde{W}]_H$.

Exercise 3. Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. Consider a continuous curve Y_t in T_eG with $t \in [0, 1]$. Show that there exist a unique curve a_t in G of class C^1 such that $a_0 = e$ and $\dot{a}_t a_t^{-1} = Y_t$ for all $t \in [0, 1]$.

(please turn)

Exercise 4. Consider S^3 as the set of unit vectors in $\mathbb{C}^2$. By abuse of notation let $\pi\colon S^3 \to S^2 = \mathbb{C}\mathrm{P}^1$ be the restriction of the projection $\pi\colon \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}\mathrm{P}^1$ to S^3.

a) Show that $\pi\colon S^3 \to S^2$ is a principal S^1-bundle (called the **Hopf bundle**).

b) Consider S^1 as the unit circle in $\mathbb{C}$ with Lie algebra $i\mathbb{R}$ and exponential map $\exp(Y) = e^{iy}$ where $Y = iy \in i\mathbb{R}$. Define 1-forms on S^3 with values in $\mathbb{C}$ by

$$\alpha_j(X_0, X_1) = X_j, \quad \overline{\alpha}_j(X_0, X_1) = \overline{X}_j$$

by using the identification

$$T_{(z_0,z_1)}S^3 = \big\{(X_0, X_1) \in \mathbb{C}^2 \,\big|\, \mathcal{R}(\overline{z}_0 X_0 + \overline{z}_1 X_1) = 0\big\}$$

where $\mathcal{R}(\beta)$ denotes the real part of $\beta \in \mathbb{C}$. Show that the 1-form on S^3

$$A_{(z_0,z_1)} = \frac{1}{2}\big(\overline{z}_0\alpha_0 - z_0\overline{\alpha}_0 + \overline{z}_1\alpha_1 - z_1\overline{\alpha}_1\big)$$

has values in $i\mathbb{R}$ and is a connection 1-form for the Hopf bundle.

Hand in: during the exercise classes.