Fall term 2019

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Mathematical Gauge Theory I

Sheet 2

Exercise 1. Suppose $\pi\colon P \to M$ is a principal G-bundle and let $f\colon N \to M$ be a smooth map. Define the **pullback** of P under f to be the space

$$f^*P\colon = \{(x,p) \in N \times P |\, f(x) = \pi(p)\}$$

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a) Show that the map

$$\begin{aligned} \pi' \colon f^*P &\to N \\ (x,p) &\mapsto x \end{aligned}$$

defines a principal G-bundle.

b) Let $W \subset M$ be an embedded submanifold. Show that the restriction $\pi\colon \pi^{-1}(W) \to W$ is a well defined principal G-bundle.

c) Prove that the bundle f^*P is trivial if f is a constant map.

d) Prove that the bundle f^*P is trivial if P is trivial.

Exercise 2. Define the Möbious strip M to be the submanifold

$$M = \left\{(e^{i\theta}, re^{i\theta/2}) \in S^1 \times \mathbb{C} |\, \theta \in [0,2\pi],\, r \in [-1,1]\right\}$$

and let $\pi\colon M \to S^1$ be the projection on the first factor.

a) Show that $\pi\colon M \to S^1$ is a fibre bundle with fibre $[-1,1]$.

b) Prove that the boundary ∂M is connected and that the bundle $\pi\colon M \to S^1$ is not trivial.

c) Prove that the image of any smooth section $s\colon S^1 \to M$ intersects the zero section $e^{i\theta} \mapsto (e^{i\theta}, 0)$.

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(please turn)

Exercise 3. Let $\pi\colon M \to S^1$ be the fibre bundle from Exercise 2 and consider the maps

$$\begin{aligned} f_n\colon S^1 &\to S^1 \\ e^{i\theta} &\mapsto e^{in\theta} \end{aligned}$$

for $n \in \mathbb{Z}$.

a) Show that the pull-back bundle f_n^*M is isomorphic to the bundle $\pi_n\colon M_n \to S^1$ defined by

$$M_n = \big\{(e^{i\theta}, re^{in\theta/2}) \in S^1 \times \mathbb{C} \,|\, \theta \in [0, 2\pi],\, r \in [-1, 1]\big\},$$

where π_n is the projection on the first factor.

b) For which $n \in \mathbb{Z}$ is the pullback bundle f_n^*M trivial?

Exercise 4.

a) Let $\pi\colon E \to M$ be a fiber bundle such that the base M and the fibre F are connected. Show that E is connected.

b) Show that the $\mathrm{SO}(n)$ principal bundle $\pi\colon\ \mathrm{SO}(n+1) \to S^n$ is the bundle of oriented orthonormal frames of the tangent bundle TS^n.

b) Use part a) to show that the group $\mathrm{SO}(n)$ is connected for all n.

Hand in: during the exercise classes.