LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Mathematical Gauge Theory I

Sheet 1

Exercise 1. Let G be a Lie group. Show that the Lie bracket $[X, Y]$ of two left-invariant vector fields X, Y is a left-invariant vector field.

Exercise 2. Consider the 3-dimensional sphere S^3 as the set of unit quaternions, i.e.

$$S^3 = \{a + ib + jc + kd \in \mathbb{H} |\, a^2 + b^2 + c^2 + d^2 = 1\}.$$

Show that S^3 is a Lie group.

Exercise 3. Consider the Lie group $SL(2, \mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

a) Compute $\operatorname{tr}(\exp X)$ for $X \in \mathfrak{sl}(2, \mathbb{R})$.

b) Show that the exponential map $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to SL(2, \mathbb{R})$ is not surjective.

Exercise 4. Let G be a connected Lie group.

a) Show that if $H \subset G$ is an open subgroup then $H = G$.

b) Let $U \subset G$ an open neighbourhood of the identity e. Prove that the set $W = \bigcup_{n=1}^{\infty} U^n$ contains an open subgroup of G. Deduce that $W = G$.

c) Show that every group element $g \in G$ is of the form $g = \exp X_1 \cdot \exp X_2 \cdots \exp X_n$ for finitely many vectors $X_1, \ldots, X_n$ in the Lie algebra $\mathfrak{g}$ of G.

d) Let $\phi, \psi\colon G \to K$ be Lie group homomorphisms. Show that if $\phi_* = \psi_*\colon \mathfrak{g} \to \mathfrak{h}$ then $\phi = \psi$.

Hand in: during the exercise classes.