Fall term 2018

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Topology I

Sheet 12

Exercise 1. For any group G, define the commutator subgroup $G' \subseteq G$ to be the subgroup generated by the elements $ghg^{-1}h^{-1}$ for all $g, h \in G$. The abelianization of G is defined as $G^{ab} := G/G'$.

Let X be a path-connected topological space and $x_0 \in X$ be a point. Show that the map

$$\varphi_1 : \pi_1(X, x_0) \longrightarrow H_1(X)$$

induces an isomorphism $\pi_1(X, x_0)^{ab} \cong H_1(X)$.

Exercise 2. For an abelian group A set $\operatorname{rk} A = \dim A \otimes \mathbb{Q}$. Let X, Y be path connected topological spaces s.t. $\operatorname{rk} H_1(X)$ and $\operatorname{rk} H_1(Y)$ are finite and $p : X \longrightarrow Y$ a covering map with finitely many sheets. Using the previous exercise, show that there is an inequality $\operatorname{rk} H_1(Y) \leq \operatorname{rk} H_1(X)$.

Exercise 3.

a) Let $\varphi : (A_\bullet, d) \longrightarrow (B_\bullet, d')$ be a map of long exact sequences s.t. φ_n is an isomorphism whenever $3|n$. Show that there is a long exact sequence of the form

$$... \longrightarrow A_{n+2} \longrightarrow A_{n+1} \oplus B_{n+2} \longrightarrow B_{n+1} \longrightarrow A_{n-1} \longrightarrow ...$$

b) Let X be a topological space and $U, V \subseteq X$ s.t. $X = \mathring{U} \cup \mathring{V}$. Show that for any sequence of functors $\widetilde{H}_n$ from pairs of topological spaces to abelian groups satisfying the Eilenberg-Steenrod axioms there is a long exact sequence

$$... \longrightarrow \widetilde{H}_n(U \cap V) \longrightarrow \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_{n-1}(U \cap V) \longrightarrow ...$$

Exercise 4. Using the previous exercise and assuming the Eilenberg-Steenrod axioms for $H_\bullet$, compute $H_k(S^n)$ for all k and n.

Hand in: during the lecture on Tuesday, January 22nd.