Fall term 2018

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Topology I

Sheet 11

Exercise 1.

a) Consider the chain complex (C_*, ∂) with $C_n = \mathbb{Z}$ for all $n \in \mathbb{Z}_{\geq 0}$ and $C_n = 0$ for $n < 0$. Let ∂_{2n} be multiplication by 2 and $\partial_{2n+1} = 0$ for $n \in \mathbb{Z}_{\geq 0}$. Compute the homology groups $H_i(C_*)$ for all $i \in \mathbb{Z}$.

b) Consider a chain complex (D_*, ∂) such that $D_n = 0$ for n odd. What are the homology groups $H_i(D_*)$?

c) In the definition of a chain complex (C_*, ∂) the groups C_n are abelian. Why are they not arbitrary groups?

Exercise 2. Let (C_*, ∂^C) and (D_*, ∂^D) be chain complexes. Denote by G_n the group of chain maps $f : C_* \to D_{*+n}$ of degree n. Show that (G_*, ∂^G) is a chain complex where ∂^G is defined by $\partial^G f(c) = f(\partial^C c)$.

Exercise 3. Let (C_*, ∂) be a chain complex such that C_n is a vector space and ∂_n is a linear map for all n.

a) Show that (C_*, ∂) splits as a direct sum of chain complexes of the following form:

   i) $\cdots \to 0 \to 0 \to V \to V \to 0 \to 0 \to \cdots$ where the map between the non trivial spaces is an isomorphism,

   ii) $\cdots \to 0 \to 0 \to W \to 0 \to 0 \to \cdots$.

b) Show that only summands of type *ii)* with W in degree n contribute to the homology group $H_n(C_*)$.

c) Give an example of an arbitrary chain complex (C_*, ∂) (i.e. the groups C_n are not vector spaces) that does not satisfy the previous properties.

(please turn)

Exercise 4. Let (C_*, ∂) be a chain complex such that C_n is a vector space and ∂_n is a linear map for all n. Assume moreover that C_n is finite dimensional for all n and $C_n \neq 0$ only for finitely many n. Prove that

$$\sum_{i\in\mathbb{Z}}(-1)^i \dim C_i = \sum_{i\in\mathbb{Z}}(-1)^i \dim H_i(C_*).$$

Hand in: during the lecture on Tuesday, January 15th.