LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
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MATHEMATISCHES INSTITUT

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Topology I

Sheet 10

Exercise 1. Let X be the two-sphere with north and south pole identified. Equip this with the structure of a CW-complex and compute the fundamental group.

Exercise 2. Let (X, A) be a pair of spaces. Write

$$C(A) := A \times [0,1]/\sim \text{ where } (a,1) \sim (a',1)\ \forall a, a' \in A$$

for the cone over A and $f : A \longrightarrow C(A)$ for the inclusion into $A \times \{0\}$.

a) If (X, A) satisfies the homotopy extension property, show that $X \cup_f C(A)$ is homotopy equivalent to X/A.

b) Let $X = [0,1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$. Show that (X, A) does not satisfy the homotopy extension property.

Exercise 3. Consider the sequence of maps

$$\mathbb{R} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \longrightarrow ...$$

where $\mathbb{R}^i \longrightarrow \mathbb{R}^{i+1}$ is the inclusion $(x_1, ..., x_n) \longmapsto (x_1, ..., x_i, 0)$ to the hyperplane $\{x_{i+1} = 0\}$ and let $\mathbb{R}^\infty$ be the union of all $\mathbb{R}^i$, equipped with the weak topology. Let S^∞ be the space defined as $S^\infty = \{x \in \mathbb{R}^\infty | \sum_{n=1}^\infty x_n^2 = 1\}$ with the subspace topology.

a) Show that the map

$$\begin{aligned} H : S^\infty \times [0,1] &\longrightarrow S^\infty \\ (x_1, x_2, \dots) &\longmapsto \frac{((1-t)x_1, tx_1 + (1-t)x_2, tx_2 + (1-t)x_3, \dots)}{N} \end{aligned}$$

where N is the norm of the point at the numerator, is a well defined homotopy.

b) Show that the map

$$\begin{aligned} \widetilde{H} : \qquad A \times [0,1] &\longrightarrow S^\infty \\ (0, x_2, x_3, \dots) &\longmapsto \frac{(t, (1-t)x_2, (1-t)x_3, \dots)}{N} \end{aligned}$$

where $A = \{x \in S^\infty | x_1 = 0\}$, defines a homotopy and conclude that S^∞ is contractible.

(please turn)

Exercise 4. This is a continuation of the previous exercise.

a) Equip S^∞ with the structure of a CW-complex.

b) Give a definition of $\mathbb{R}\mathrm{P}^\infty$, s.t. there is a two-sheeted covering map $S^\infty \longrightarrow \mathbb{R}\mathrm{P}^\infty$.

c) Show that the spaces $\mathbb{R}\mathrm{P}^2$ and $\mathbb{R}\mathrm{P}^\infty \times S^2$ have the same homotopy groups.

Remark One can show that $\mathbb{R}\mathrm{P}^2$ and $\mathbb{R}\mathrm{P}^\infty \times S^2$ are not homotopy equivalent. Why is this not a contradiction to Whitehead's theorem?

Hand in: during the lecture on Tuesday, January 8th.