Fall term 2018

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Topology I

Sheet 6

Exercise 1. Show that the projection $S^n \longrightarrow \mathbb{RP}^n$ is a two-sheeted covering map. Compute the fundamental group of $\mathbb{RP}^n$ for any $n \in \mathbb{Z}_{>0}$.

Exercise 2. Let $C := [0,1] \times [0,1]/(0,y) \sim (1,y)$ be the cylinder and $M := [0,1] \times [0,1]/(0,y) \sim (1,1-y)$ be the Möbius band. For any $n \in \mathbb{Z}_{>0}$, show that:

i) There is a covering map $C \to C$ with n sheets.

ii) There is a covering map $C \longrightarrow M$ with $2n$ sheets.

iii) There is a covering map $M \longrightarrow M$ with $2n-1$ sheets.

iv) The maps $\partial M \longrightarrow [0,1]/\{0\} \sim \{1\}$ and $\partial C \longrightarrow [0,1]/\{0\} \sim \{1\}$, induced by projection to the first factor, are covering maps with 2 sheets.

Exercise 3. Let G be a (discrete) group and X a topological space with an action of G. The action is called a covering space action (or sometimes properly discontinuous action) if the following condition is satisfied: For any $x \in X$, there exists an open set $x \in U \subset X$, s.t. for all $g \in G$, $gU \cap U \neq \emptyset$ implies $g = e_G$.

a) Show that an action is a covering space action if, and only if, the quotient map $X \longrightarrow X/G$ is a covering map and the stabilizer subgroup $G_x = \{g \in G | gx = x\}$ is trivial for all $x \in X$.

b) If X is simply connected, i.e. $\pi_0(X) = \pi_1(X) = 1$, what is the fundamental group of X/G?

(please turn)

Exercise 4. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half-plane $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ via fractional linear transformations, i.e.:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := \frac{az+b}{cz+d}$$

i) Show that there exist points which have non-trivial stabilizer.

ii) For any N, let

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N,\ c \equiv 0 \bmod N \right\}.$$

Determine an N such that the induced action of $\Gamma_1(N)$ is a covering space action.

Hand in: during the lecture on Tuesday, November 27th.