Fall term 2018

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Topology I

Sheet 2

Exercise 1. Show that the interval $I = [0, 1] \subseteq \mathbb{R}$ is compact, without using the fact that subspaces of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.

Exercise 2. Let X, Y be topological spaces. Show that the following statements are equivalent:

1. The spaces X and Y are both compact.
2. The topological sum (disjoint union) $X + Y$ is compact.
3. The direct product $X \times Y$ is compact.

Exercise 3.

a) Show that, for any $n \in \mathbb{Z}_{\geq 0}$, the space $\mathbb{R}^{n+1} \setminus \{0\}$ is homotopy equivalent to the sphere S^n.

b) Show that star-shaped subspaces of $\mathbb{R}^n$ are contractible. (A subspace $A \in \mathbb{R}^n$ is called **star-shaped** if there is a point $x_0 \in A$ s.t. for any point $x \in A$ the segment $\{x_0 + t(x - x_0) \mid t \in [0, 1]\}$ lies in A.)

Exercise 4. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $n \in \mathbb{Z}_{>0}$. Projective n-space $\mathbb{K}\mathrm{P}^n$ (the space of lines in $\mathbb{K}^{n+1}$) is defined as the quotient of $\mathbb{K}^{n+1} \setminus \{0\}$ by the multiplicative action of $\mathbb{K}^*$, equipped with the quotient topology. Show that

a) $\mathbb{R}\mathrm{P}^1$ is homeomorphic to the circle S^1.

b) $\mathbb{C}\mathrm{P}^1$ is homeomorphic to the sphere S^2.

Remark: These are actually the only two cases in which $\mathbb{K}\mathrm{P}^n$ is homeomorphic (or even homotopy-equivalent) to a sphere. We might see this later on.

Hand in: during the lecture on Tuesday, October 30th.