Fall term 2017

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Topology I

Sheet 11

Exercise 1. Let G be a group and $\mathcal{S}_n$ be the group of permutations of n elements. A homomorphism $f : G \to \mathcal{S}_n$ is called transitive if for all $i, j \in \{1, \dots, n\}$ there exists $g \in G$ with $f(g)(i) = j$.
Let $p : \widetilde{X} \to X$ be a n-sheeted covering. Fix $x_0 \in X$ and a numbering $\tilde{x}_1, \dots, \tilde{x}_n$ of $p^{-1}(x_0)$. For $[\alpha] \in \pi_1(X, x_0)$ define a permutation $\bar{\alpha} \in \mathcal{S}_n$ as $\bar{\alpha}(i) = j$ if $\tilde{\alpha}_i(1) = \tilde{x}_j$ where $\tilde{\alpha}_i$ is the lift of α starting at $\tilde{x}_i$.

a) Show that the map $[\alpha] \mapsto \bar{\alpha}$ gives a well defined homomorphism $f : \pi_1(X, x_0) \to \mathcal{S}_n$ which is transitive if $\widetilde{X}$ is path-connected.

b) Two such homomorphisms $f, f' : G \to \mathcal{S}_n$ are called equivalent if there exists $\sigma \in \mathcal{S}_n$ such that $f'(g) = \sigma f(g) \sigma^{-1}$ for all $g \in G$. Show that the equivalence class $\langle f \rangle$ of $f : \pi_1(X, x_0) \to \mathcal{S}_n$ does not depend on the chosen numbering for $p^{-1}(x_0)$. Since it only depends on the covering denote $\langle f \rangle$ by $\mathcal{S}(\widetilde{X}, p)$.

Exercise 2. Let X be a sufficiently connected space with basepoint $x_0 \in X$. Show that two n-sheeted, path-connected coverings $p : \widetilde{X} \to X$, $p' : \widetilde{X}' \to X$ are equivalent if and only if $\mathcal{S}(\widetilde{X}, p) = \mathcal{S}(\widetilde{X}', p')$.

Exercise 3. Let $\langle G, \mathcal{S}_n \rangle$ be the set of equivalence classes of transitive homomorphisms $f : G \to \mathcal{S}_n$. Prove that for every element $a = \langle \varphi \rangle \in \langle \pi_1(X, x_0), \mathcal{S}_n \rangle$ there is a n-sheeted, path-connected covering $p : \widetilde{X} \to X$ with $\mathcal{S}(\widetilde{X}, p) = a$. Conclude that there is a $1-1$ correspondence between n-sheeted coverings of X and elements of $\langle \pi_1(X, x_0), \mathcal{S}_n \rangle$.
[Hint: Consider the subgroup H of $\pi_1(X, x_0)$ given by $H = \{\alpha \in \pi_1(X, x_0) | \varphi(\alpha)(1) = 1\}$]

Exercise 4. Classify and draw all 2-sheeted coverings of $S^1 \vee S^1$.

Hand in: during the lecture on Monday, January 15th.