Fall term 2017

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Topology I

Sheet 9

Exercise 1.

a) Show that if $pr_1 : \widetilde{X_1} \to X_1$ and $pr_2 : \widetilde{X_2} \to X_2$ are covering maps, then $pr_1 \times pr_2 : \widetilde{X_1} \times \widetilde{X_2} \to X_1 \times X_2$ is a covering map.

b) Show that if $pr : Y \to X$ is a covering map and $A \subset X$ then $pr : pr^{-1}(A) \to A$ is a covering map.

Exercise 2. Let X be the space given by the union of the graph $\left\{ \left(x, \sin(\frac{1}{x})\right) | 0 < x \leq \frac{1}{\pi} \right\}$, the segment $Y = \{(0, y) | -1 \leq y \leq 1\}$ and a path connecting the points $(0,0)$ and $(\frac{1}{\pi}, 0)$.

a) Prove that X is simply connected.

b) Show that the quotient X/Y is homeomorphic to S^1.

c) Show that the quotient map $p : X \to S^1$ does not admit a lift $\tilde{p} : X \to \mathbb{R}$ (i.e. the local path-connectedness of X is a necessary hypothesis in the lifting criterion).

Exercise 3. Let (X, d) be a metric space. Define the length $L(\gamma)$ of a curve γ to be

$$L(\gamma) := \lim_{n\to\infty} \sum_{1=0}^{n} \mathrm{d}(\gamma(t_i), \gamma(t_{i-1}))$$

where the t_i's give a regular partition of $[0,1]$. (X, d) is a **length space** if $\mathrm{d}(x_0, x_1) = \inf_\gamma L(\gamma)$ where the infimum is taken over all paths γ connecting x_0 to x_1.
Suppose $pr : Y \to X$ is a covering such that Y is path connected and X is a length space.

a) Prove that there exists a metric d_Y for which Y is a length space.

b) Show that for any $y \in Y$ there exists a neighbourhood V of y such that $\mathrm{d}_Y(y_0, y_1) = \mathrm{d}(pr(y_0), pr(y_1))$ for all $y_0, y_1 \in V$.

(please turn)

Exercise 4. Consider the sphere S^3 as the space of unit quaternions $\{x \in \mathbb{H} | \|x\| = 1\}$ and identify $\mathbb{R}^3$ with the pure imaginary quaternions $\{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} | a = 0\}$. Prove that the map

$$\begin{aligned} pr : S^3 \to &\quad SO(3) \\ x \mapsto &\ (p \mapsto xpx^{-1}) \end{aligned}$$

with multiplication in $\mathbb{H}$, is a well defined covering map.

Hand in: during the lecture on Monday, December 18th.