Fall term 2017

Prof. T. Vogel
G. Placini

Topology I

Sheet 5

Exercise 1. Let X be a topological space. Show that the following are equivalent:

i) X is contractible.

ii) For every Y and $f : Y \to X$, f is nullhomotopic.

iii) For every Y and $f : X \to Y$, f is nullhomotopic.

Exercise 2.

a) Show that there exists a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}.

b) Consider the space $X = \bigcup_{n=1}^{\infty} X_n$ where $X_n = \{(x, nx) \in \mathbb{R}^2 | 0 \leq nx \leq 1\}$ for $n \in \mathbb{N}$ while $X_\infty = \{(0, y) \in \mathbb{R}^2 | 0 \leq y \leq 1\}$. Prove that X deformation retracts to $(0, 0)$ but not to $(0, 1)$.

Exercise 3.

a) Let $f : D^2 \to \mathbb{R}^2$ be a continuous map such that $f(-x) = -f(x)$ for all $x \in S^1$. Show that there exists $y \in D^2$ such that $f(y) = 0$.

b) Show that the following system of equalities admits a solution

$$\begin{cases} x\cos(y) = x^2 + y^2 - 1 \\ y\cos(x) = \sin 2\pi(x^2 + y^2) \end{cases}$$

Exercise 4. Let A,$B \subset S^1$ be closed sets such that $A \cup B = S^1$. Show that A or B contains a pair of antipodal points.

[Hint: Define a suitable function $f : S^1 \to \mathbb{R}$ and use the intermediate value theorem.]

(please turn)

Exercise 5.

a) Let L be the CW-complex with a single 0-cell v_0 and a countable family $b_j, j \in \mathbb{N}$ of 1-cells such that the characteristic maps F_j satisfy image$(f_j) = F_j(\{0,1\}) = v_0$. Each 1-cell has a coordinate $x_j \in [0,1]$ such that $0, 1$ corresponds to v_0.

Let V be a neighbourhood of v_0 in L. Show that there is a sequence $\delta_j \in (0, 1/2)$ such that

$$V_{(\delta_j)} = \{v_0\} \cup \bigcup_{i \in \mathbb{N}} (b_j \setminus [\delta_j, 1 - \delta_j])$$

is contained in V.

Now let K be the analogous complex, with the 0-cell u_0 and the 1-cells a_i indexed by the set

$$I = \{(i_1, i_2, i_3, \ldots) | i_j \in \{2, 3, 4, \ldots\}\}$$

of integer sequences ($i \in I$ denotes such a sequence). Recall that I is uncountable.

b)Every neighbourhood of u_0 contains a set of the form

$$U_\varepsilon = \{u_0\} \cup \bigcup_{i \in I} (a_i \setminus [\varepsilon_i, 1 - \varepsilon_i]).$$

c) for $i \in I$ and $j \in \mathbb{N}$ let $p_{i,j} = (1/j, 1/i_j) \in a_i \times b_j \subset K \times L$. Show that if we equip $K \times L$ with the product cellular decompositions and characteristic maps obtained from products of characteristic maps of K and L then $K \times L$ is a locally finite cell complex and we equip it with the weak topology. We denote $K \times L$ with this topology by $(K \times L)_W$ (i.e. a set in $(K \times L)_W$ is closed iff its intersection with every closed cell is closed). Show that $P = \{p_{i,j} | i \in I, j \in N\}$ is closed in $(K \times L)_W$.

d) Finally, show that every neighbourhood of $(u_0, v_0) \in K \times L$ with the product topology contains a point of P. Since $(u_0, v_0) \notin P$ this implies that P is **not** closed in the product topology.

[Every neighbourhood of (u_0, v_0) contains a set of the from $U_\varepsilon \times V_\delta$ as above. Choose $\bar{i} = (\bar{i}_1, \bar{i}_2, \ldots) \in I$ such that $\bar{i}_j > j$ and $\bar{i}_j > \delta_j^{-1}$. Choose $\bar{j} > \varepsilon_{\bar{i}}^{-1}$ and show that $p_{\bar{i},\bar{j}} \in U_\varepsilon \times V_\delta$.]

Hand in: during the lecture on Monday, November 20th.