Fall term 2017

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Topology I

Sheet 2

Exercise 1. Prove the following statements.

a) Any subspace of a Hausdorff space is Hausdorff.

b) An arbitrary product of Hausdorff spaces is Hausdorff.

c) $\mathbb{R}^n/\mathring{D}^n$ is not Hausdorff.

d) $\mathbb{R}^n/\overline{D^n}$ is homeomorphic to $\mathbb{R}^n$.

Exercise 2.

a) Show that the space $S^n \times S^m/(S^n \vee S^m)$ is homeomorphic to S^{n+m}.
[Hint: You can start by showing that $S^n \times S^m \setminus (S^n \vee S^m) \simeq D^{n+m}$ and use then the homeomorphism $S^n \simeq D^n/\partial D^n$.]

b) Prove that the suspension $\sum(S^n)$ is homeomorphic to S^{n+1}.

Exercise 3.

a) Let $\{X_\alpha\}_{\alpha \in A}$ be a family of topological spaces. Describe the finest topology on the disjoint union $\dot{\bigcup}_{\alpha \in A} X_\alpha$ such that the injections $i_\beta : X_\beta \to \dot{\bigcup}_{\alpha \in A} X_\alpha$ are continuous.

b) Let $X_1, \ldots, X_n$ be topological spaces and x_i a point in X_i for $1 \le i \le n$. Show that the natural map, given by

$$\begin{aligned} X_1 \vee \cdots \vee X_n &\longrightarrow X_1 \times \cdots \times X_n \\ y_i \quad &\mapsto (x_1, \ldots, y_i, \ldots, x_n) \end{aligned}$$

for $y_i \in X_i$, is continuous.

c) Set $X_i = S^1$ for all $i \in \mathbb{N}$, $x_i = 1 \in S^1$ and consider $X = \bigvee_{i \in \mathbb{N}} X_i$. Is the map

$$f : X \to \prod_{i \in \mathbb{N}} X_i,$$

analogous to point b), a homeomorphism onto its image?
[Hint: You may use that $\prod_{i \in \mathbb{N}} X_i$ is compact.]

Exercise 4. Show that the complex projective space $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2.
[Hint: Consider $\mathbb{C}^2 = \{(x, y) \in \mathbb{C}^2 | y \neq 0\} \cup \{(x, 0) \in \mathbb{C}^2\}$ and their image under the quotient.]

(please turn)

Exercise 5. [1] A topological space Y is said **1st countable** if for each point $y \in Y$ there exists a family of open neighbourhoods $\{U_j\}_{j\in\mathbb{N}}$ such that for any neighbourhood V of y there exist $j \in \mathbb{N}$ with $U_j \subset V$.

a) Show that if $\{X_i\}_{i\in\mathbb{N}}$ is a family of 1^{st} countable spaces then so is $\prod_{i\in\mathbb{N}} X_i$.

b) Prove that $\bigvee_{i\in\mathbb{N}} S^1$ is not 1^{st} countable.

c) Conclude that there is no embedding

$$f : \bigvee_{i\in\mathbb{N}} S^1 \longrightarrow \prod_{i\in\mathbb{N}} S^1.$$

Hand in: during the lecture on Monday, October 30th.

[1] For motivated students, this exercise is harder but it complements exercise 3.