Summer term 2020

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Mathematical Gauge Theory II

Sheet 12

Exercise 1. (Complex conjugation on $\mathbb{CP}^2$) We want to show that there exists an orientation preserving diffeomorphism $d\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ which is the identity on some ball $D^4 \subset \mathbb{CP}^2$ and induces $-\mathrm{Id}$ on $H_2(\mathbb{CP}^2;\mathbb{Z})$.

1. Consider the map $c\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ given by complex conjugation of the homogeneous coordinates. Prove that c is orientation preserving and induces $-\mathrm{Id}$ on $H_2(\mathbb{CP}^2;\mathbb{Z})$.
2. Show that c preserves $\mathbb{C}^2 = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid z_0 = 1\}$ and find an explicit isotopy f_t on $\mathbb{C}^2$ with $f_0 = \mathrm{Id}_{\mathbb{C}^2}$ and $f_1 = c^{-1}|_{\mathbb{C}^2}$.
3. Let $D^4 \subset \mathbb{C}^2$ be a closed ball. Prove that c is isotopic to an orientation preserving diffeomorphism $d\colon \mathbb{CP}^2 \to \mathbb{CP}^2$ with $d|_{D^4} = \mathrm{Id}_{D^4}$.

Exercise 2. (Reflection in (±1)-sphere)

1. Let N be a smooth oriented 4-manifold and $M = N\#\mathbb{CP}^2$ or $M = N\#\overline{\mathbb{CP}}^2$. Let $E \in H_2(M;\mathbb{Z})$ be the homology class of the sphere $\mathbb{CP}^1 \subset \mathbb{CP}^2 \setminus D^4 \subset M$ with self-intersection $E^2 = \pm1$. Use the diffeomorphism d from Exercise 1 to show that there exists an orientation preserving diffeomorphism $f\colon M \to M$ which induces on integer homology the map f_* given by
$$\begin{aligned} f_*\colon H_2(M;\mathbb{Z}) &\longrightarrow H_2(M;\mathbb{Z}) \\ A &\longmapsto A \mp 2(A\cdot E)E. \end{aligned}$$
2. For an arbitrary smoothly embedded sphere S^2 of self-intersection ±1 in a 4-manifold X, show that a tubular neighbourhood is diffeomorphic to a punctured $\mathbb{CP}^2$, and conclude that X is diffeomorphic to $Y\#\mathbb{CP}^2$ or $Y\#\overline{\mathbb{CP}}^2$, so that the above result is applicable.

Exercise 3. (Spheres with self-intersection zero) Suppose that Y is a smooth closed oriented 4-manifold with $b_2^+(Y) \geq 2$ which contains a smoothly embedded $S^2 \hookrightarrow Y$ of self-intersection zero, representing a class of infinite order in $H_2(Y;\mathbb{Z})$. Consider the manifold $X = Y\#\overline{\mathbb{CP}}^2$.

1. Prove that there exist infinitely many pairwise distinct classes $S_i \in H_2(X;\mathbb{Z})$ represented by embedded 2-spheres of self-intersection -1.
2. Show that if there exists a Spin^c-structure $\mathfrak{s}$ on X with non-zero Seiberg-Witten invariant, then there are infinitely many such structures. Conclude that the Seiberg-Witten invariants of X and Y are in fact identically zero.

Hint: You may use Theorem 8.31.

(please turn)

Exercise 4. (Seiberg-Witten invariants of $p\mathbb{CP}^2\#q\overline{\mathbb{CP}}^2$) Let $X = p\mathbb{CP}^2\#q\overline{\mathbb{CP}}^2$. Suppose that $p, q \geq 2$. Prove that $SW_X \equiv 0$.

Hint: This appeared in Example 8.29, as a consequence of Theorem 8.28. It also follows from the fact that X admits metrics with positive scalar curvature. You should give a proof which is independent of these results, using Exercise 3 above.

You can email the solutions until Tuesday, July 21st at noon.