Summer term 2020

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Mathematical Gauge Theory II

Sheet 11

Exercise 1. (Unperturbed SW equation on $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$) Consider $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ endowed with a metric with positive scalar curvature as in Exercise 1 from Sheet 10.

(a) Classify Spin^c-structures on $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ in terms of the cohomology.

(b) Compute the expected dimension of the moduli space of solutions to the unperturbed Seiberg-Witten equation for any Spin^c-structure.

(c) Prove that for every Spin^c-structure the unperturbed Seiberg-Witten equation has no solution.

Exercise 2. (Small perturbations of the Seiberg-Witten equations on T^4 II) As in Exercise 3 from Sheet 10 consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. For a Spin^c-structure $\mathfrak{s}$ on T^4 consider the perturbed Seiberg-Witten equations

$$\begin{aligned} D_A^+\Phi &= 0 \\ F_{\hat{A}}^+ &= \sigma(\Phi,\Phi) + i\varepsilon\omega \ , \end{aligned}$$

where $0 < \varepsilon << 1$ is real and positive, and very small. Assume that the expected dimension of the moduli space of solutions is non-negative. Prove that $c_1(L_{\mathfrak{s}}) = 0$, as soon as there is a solution.

Exercise 3. (Some more integral bounds) Let (X, g) be a closed, oriented, connected, Riemannian 4-manifold. For any solution (A,Φ) to the unperturbed Seiberg-Witten equation with non-negative expected dimension prove the following integral bounds:

$$\begin{aligned} \left\| F_{\hat{A}}^+ \right\|_{L^2}^2 &\leq \frac{1}{8} \left\| s_{g,0} \right\|_{L^2}^2 \ , \\ \left\| F_{\hat{A}}^- \right\|_{L^2}^2 &\leq \frac{1}{8} \left\| s_{g,0} \right\|_{L^2}^2 - 8\pi^2\chi(X) - 12\pi^2\sigma(X) \ . \end{aligned}$$

What are the analogous bounds for parameters (g,ω)?

(please turn)

Exercise 4. (Even intersection forms and spin manifolds) Let X be a closed, oriented, connected, smooth 4-manifold without 2-torsion in $H^2(X;\mathbb{Z})$. Recall that a **characteristic** element for Q_X is an element $c \in H^2(X;\mathbb{Z})$ which satisfies

$$Q_X(c, a) \equiv Q_X(a, a) \mod 2$$

for all $a \in H^2(X;\mathbb{Z})$.

(a) Let $L_{\mathfrak{s}}$ be the characteristic line bundle of a Spin^c-structure $\mathfrak{s}$. Use the Atiyah Index Theorem to show that $c_1(L_{\mathfrak{s}})$ is a characteristic element for Q_X.

(b) Show that any characteristic element $c \in H^2(X;\mathbb{Z})$ satisfies $c = c_1(L_{\mathfrak{s}})$ for some Spin^c-structure $\mathfrak{s}$ on X.

(c) Conclude that a closed, oriented, connected, smooth 4-manifold X without 2-torsion in $H^2(X;\mathbb{Z})$ is spin if and only if its intersection form Q_X is even.

You can email the solutions until Tuesday, July 14th at noon.