Summer term 2020

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Mathematical Gauge Theory II

Sheet 10

Exercise 1. (Metric of positive scalar curvature on $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$) Prove that $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ has a metric with positive scalar curvature, without using the hint from Exercise 4 of Sheet 9.

Exercise 2. (Seiberg-Witten equations on $\mathbb{C}\mathrm{P}^2$) Let $z = (z_1, z_2)$ be local coordinates on $\mathbb{C}\mathrm{P}^2$. Consider the Fubini-Study metric

$$g_{FS} = \sum_{i,j=1}^{2} \frac{\delta_{ij}(1+|z|^2) - \overline{z}_i z_j}{(1+|z|^2)^2} \,\mathrm{d}\, z_i \otimes \mathrm{d}\, \overline{z}_j$$

associated to the Kähler form

$$\omega_{FS} = i\partial\overline{\partial}\log(1+|z|^2).$$

Note that ω_{FS} is a parallel 2-form.

(a) Classify Spin^c-structures on $\mathbb{C}\mathrm{P}^2$ in terms of the cohomology.

(b) Show that the Fubini-Study metric has positive scalar curvature.

(c) Prove that for every Spin^c-structure the unperturbed Seiberg-Witten equation has no solution.

(d) Consider the perturbed Seiberg-Witten equations

$$D_A^+\Phi = 0$$
$$F_{\hat{A}}^+ = \sigma(\Phi, \Phi) + i\varepsilon\omega_{FS} \;.$$

Show that for every Spin^c-structure there is a unique ε such that the equations have precisely one solution, which is reducible. What is the relation between this value of ε and the Spin^c-structure?

(please turn)

Exercise 3. (Small perturbations of the Seiberg-Witten equations on T^4) Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Note that this is a parallel g_0-self-dual 2-form.

For a Spin^c-structure $\mathfrak{s} = (\gamma, V)$ on T^4 consider the perturbed Seiberg-Witten equations

$$\begin{aligned} D_A^+ \Phi &= 0 \\ F_{\hat{A}}^+ &= \sigma(\Phi, \Phi) + i\varepsilon\omega \ , \end{aligned}$$

where $0 < \varepsilon << 1$ is real and positive, and very small. Assume that the expected dimension of the moduli space of solutions is non-negative.

(a) Prove that if there is a solution to the equations, then $\langle c_1^2(L_{\mathfrak{s}}), [T^4] \rangle = 0$, equivalently the expected dimension is zero.

(c) For the unique Spin^c-structure with $c_1(L_{\mathfrak{s}}) = 0$ prove that there is precisely one solution up to gauge equivalence for every $\varepsilon \neq 0$.

Exercise 4. (Non-homotopic almost complex structures)

(a) Show that the space of orthogonal almost complex structures on $\mathbb{R}^4$ is $SO(4)/U(2) \cong \mathbb{C}P^1$.

(b) Show that there are almost complex structures J_0, J_1 on a suitable 4-manifold ,e.g. $S^1 \times S^3$, that have the same Chern classes but that are not homotopic as almost complex structures.

[Hint: You can use the fact that $\pi_3(S^2) = \mathbb{Z}$.]

You can email the solutions until Tuesday, July 7th at noon.