Summer term 2020 Prof. D. Kotschick
G. Placini

Mathematical Gauge Theory II

Sheet 9

Exercise 1. (Seiberg-Witten equations on the flat torus) Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Prove the following statements.

(a) Any solution (A, Φ) to the unperturbed Seiberg-Witten equations on (T^4, g_0) is reducible, i.e. Φ vanishes identically, and has flat $\hat{A}$.

(b) If the expected dimension of the moduli space for a Spin^c-structure $\mathfrak{s}$ on T^4 is non-negative, and the moduli space is non-empty, then the Spin^c-structure is the unique one induced by any spin structure (cf. Remark 2.32), and the moduli space is a copy of T^4.

Exercise 2. (The even expected dimension case) Let (X, g) be a smooth closed oriented Riemannian 4-manifold endowed with a Spin^c-structure $\mathfrak{s}$. Show that if the expected dimension of the moduli space is even, then $b_2^+(X) - b_1(X)$ is odd.

Exercise 3. (Seiberg-Witten equations on $S^2 \times S^2$) Consider $S^2 \times S^2$ with the product metric, where each factor is a round sphere, i.e. of constant curvature.

(a) Determine the moduli spaces of solutions to the unperturbed Seiberg-Witten equations for all Spin^c-structures.

(b) Conclude that whenever the moduli space is non-empty, then the expected dimension is negative.

Exercise 4. (Seiberg-Witten equations on $\#n(S^1 \times S^3)$) Consider $S^1 \times S^3$, with the product metric coming from two round factors.

(a) Show that there is a unique Spin^c-structure, and determine the moduli space of solutions to the unperturbed Seiberg-Witten equations. How does the dimension of the result compare to the expected dimension?

(b) Extend this discussion to connected sums of several copies of $S^1 \times S^3$.

[Hint: you can use the fact that the connected sum of two Riemannian manifolds with positive scalar curvature admits a metric with positive scalar curvature.]

You can email the solutions until Tuesday, June 30th at noon.