Summer term 2020

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Mathematical Gauge Theory II

Sheet 7

Exercise 1. (Non-negative scalar curvature) Let (X, g) be a smooth closed oriented Riemannian 4-manifold with non-negative scalar curvature endowed with a Spin^c-structure $\mathfrak{s}$. Show that any solution (A, Φ) to the unperturbed Seiberg-Witten equations is reducible, that is, satisfies $\Phi = 0$.

Exercise 2. (Rescaling the metric) Let (X, g) be a smooth closed oriented Riemannian 4-manifold with a Spin^c-structure $\mathfrak{s}$ and let $\omega \in \Omega^2_+(X, i\mathbb{R})$. Consider the rescaled metric $\widetilde{g} = \lambda^2 g$ for $\lambda \in \mathbb{R}^+$.

1. Prove that if (A, Φ) is a solution to the ω-perturbed Seiberg-Witten equations on (X, g), then $(A, \lambda^{-1}\Phi)$ satisfies the Seiberg-Witten equations on $(X, \widetilde{g})$.
   [Notice that the Clifford module $\gamma \colon TX \longrightarrow \mathrm{End}(V)$ is rescaled.]
2. Show that the energy of the pair (A, Φ) with respect to the metric g equals the energy of $(A, \lambda^{-1}\Phi)$ with respect to the metric $\widetilde{g}$.

Exercise 3. (Reducible solutions) Let (X, g) be a smooth closed oriented Riemannian 4-manifold endowed with a Spin^c-structure $\mathfrak{s}$. Show that the unperturbed Seiberg-Witten equations admit a reducible solution if and only if the harmonic representative of $c_1(L_{\mathfrak{s}})$ is anti-self-dual.

Exercise 4. (Chern classes of tensor products, cf. Remark 4.41 in the notes)

1. Let V, W be complex vector bundles of rank 2. Use the splitting principle to prove that

$$\begin{aligned} c_1(V \otimes W) &= 2\,(c_1(V) + c_1(W)) \\ c_2(V \otimes W) &= 2\,(c_2(V) + c_2(W)) + c_1^2(V) + c_1^2(W) + 3c_1(V)c_1(W). \end{aligned}$$

2. Let V_+ be the spinor bundle of a Spin^c-structure over an oriented Riemannian 4-manifold. Use an isomorphism induced from Clifford multiplication to prove that

$$p_1\left(\Lambda^2_+\right) = c_1^2\left(V_+\right) - 4c_2\left(V_+\right).$$

You can email the solutions until Tuesday, June 16th at noon.