Summer term 2020 Prof. D. Kotschick
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Mathematical Gauge Theory II

Sheet 6

Exercise 1. (S^2-bundles over Σ_g) Let Σ_g denote the surface of genus g.

1. Suppose $D^2 \subset \Sigma_g$ is a small disk around a point. Show that $\Sigma_g \setminus D^2$ is homotopy equivalent to a 1-point union $\bigvee_{i=1}^{2g} S_i^1$ of $2g$ circles.
2. Prove that for every $g \geq 0$ there are at most two orientable S^2-bundles over Σ_g up to diffeomorphism.

Exercise 2.

1. Prove that if $P\colon H_1 \longrightarrow H_2$ is a Fredholm operator between Hilbert spaces, then $\operatorname{coker} P \cong \ker P^*$.
2. Let M be a closed oriented connected Riemannian manifold and let d^* be the formal adjoint of d with respect to the L^2 scalar product. Show that the operator

$$P = \mathrm{d} + \mathrm{d}^*\colon \bigoplus_{k \text{ even}} \Omega^k(M) \longrightarrow \bigoplus_{k \text{ odd}} \Omega^k(M)$$

is Fredholm by showing that $\dim \ker(P)$ and $\dim \operatorname{coker}(P)$ are finite. Compute $\operatorname{ind}(P)$.

Exercise 3. (Spinor identities) Let $\Gamma(V_+)$ be the space of positive spinors associated to a $\mathrm{Spin}^c(n)$-structure on a smooth closed oriented Riemannian 4-manifold. We give $\mathrm{End}(V_+)$ the scalar product

$$\langle A, B\rangle = \operatorname{tr}\left(AB^\dagger\right)$$

and define the quadratic form σ as in the notes:

$$\begin{aligned}\sigma\colon \Gamma(V_+) &\longrightarrow \Omega^2_+(X, i\mathbb{R})\\ \Phi &\longmapsto \sigma(\Phi,\Phi) = \gamma^{-1}\left((\Phi\otimes\Phi^\dagger)_0\right).\end{aligned}$$

Prove for all $\omega, \eta \in i\Lambda^2_+$ and $\Phi \in V_+$ the following identities:

$$\begin{aligned}\langle\gamma(\omega), \gamma(\eta)\rangle &= 4\langle\omega, \eta\rangle\\ \langle\gamma(\omega)\Phi, \Phi\rangle &= 4\langle\omega, \sigma(\Phi,\Phi)\rangle\\ |\Phi|^4 &= 8|\sigma(\Phi,\Phi)|^2.\end{aligned}$$

(please turn)

Exercise 4. (The quadratic form σ and charge conjugation) We define charge conjugation on spinors in $V_+ \cong \mathbb{C}^2$ as:

$$\begin{aligned} J\colon V_+ &\longrightarrow V_+ \\ \begin{pmatrix} a \\ b \end{pmatrix} &\longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix}. \end{aligned}$$

Prove that

$$\sigma(J\Phi, J\Phi) = -\sigma(\Phi, \Phi) \quad \forall \Phi \in V_+.$$

You can email the solutions until Tuesday, June 9th at 9:00.