Summer term 2020

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Mathematical Gauge Theory II

Sheet 5

Exercise 1. (Invariants of 4-manifolds) Let M and N be two closed, connected, simply connected, oriented, smooth 4-manifolds.

1. Prove that M and N are homeomorphic if and only if the the following invariants agree:
   - Euler characteristic χ
   - signature σ
   - parity (even or odd) of the intersection form.
2. Determine a simple 4-manifold homeomorphic to $M\#\overline{\mathbb{CP}}^2$.
3. Assume $\sigma(M) = -\sigma(N)$ and even intersection forms Q_M, Q_N. Determine a 4-manifold homeomorphic to $M\#N$.

Exercise 2. (Embedded surfaces in $\mathbb{CP}^2$) A *projective line* is a linear $\mathbb{CP}^1$ in $\mathbb{CP}^2$ (coming from a linear subspace $\mathbb{C}^2 \subset \mathbb{C}^3$). Let $d \geq 0$ be a natural number.

1. We call d projective lines in $\mathbb{CP}^2$ *in general position* if all intersections between them are transverse and if at most two projective lines intersect in a given point p for all $p \in \mathbb{CP}^2$. Prove that there exists d projective lines in $\mathbb{CP}^2$ in general position for all $d \geq 0$.
2. Determine a smooth surface representing the class $d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})$. What is its genus?

Exercise 3. (The double of a manifold) Let M be a compact oriented manifold with non-empty boundary and let $\overline{M}$ denote the same manifold with the opposite orientation. The double of M is obtained by gluing together M and $\overline{M}$ along the boundary via the identity map:

$$M \bigcup_{\partial M = \partial \overline{M}} \overline{M}.$$

Now let $D(e)$ denote the disc bundle with Euler class e over S^2. Show that the double of $D(e)$ is diffeomorphic to $S^2 \times S^2$ if and and only if e is even and to $S^2 \widetilde{\times} S^2$ if and only if e is odd.

(please turn)

Exercise 4. (Complete intersections) Let $d = (d_1, d_2, \dots, d_r)$ be an r-tuple of natural numbers and consider the intersection of r smooth hypersurfaces X_{d_i} of degree d_i in $\mathbb{CP}^{r+2}$:

$$S_d = X_{d_1} \cap X_{d_2} \dots \cap X_{d_r}.$$

We assume that for all $k = 2, \dots, r$ the hypersurface X_{d_k} intersects $X_{d_1} \cap \dots \cap X_{d_{k-1}}$ transversely. Then S_d is a smooth complex surface, called a *complete intersection of multidegree d*.

1. Suppose submanifolds M and N of a manifold W intersect transversely. Show that the normal bundles in W are related by $\nu(M \cap N) = \nu(M)|_{M\cap N} \oplus \nu(N)|_{M\cap N}$.

2. Calculate the Chern classes $c_1(S_d)$ and $c_2(S_d)$.

3. Determine those multidegrees d for which S_d is a $K3$ surface.

   [Cheat: you may use without proof that all complete intersections are simply connected.]

You can email the solutions until Tuesday, May 26th at noon.