LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
MÜNCHEN

MATHEMATISCHES INSTITUT

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Mathematical Gauge Theory II

Sheet 4

Exercise 1. (Product of surfaces) The Euler characteristic of a closed n-manifold M is defined by

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i(M),$$

where $b_i(M)$ are the Betti numbers. The Euler characteristic is multiplicative for products of manifolds, that is, $\chi(M \times N) = \chi(M) \cdot \chi(N)$. Let $M = \Sigma_g \times \Sigma_h$, where Σ_g, Σ_h are surfaces of genus g and h.

1. Calculate $\chi(M)$.

2. Determine bases for all integral homology groups $H_*(M; \mathbb{Z})$ of M and calculate the Betti numbers. Compare with (1).

   [Hint: You can use the standard basis of $H_1(\Sigma_g; \mathbb{Z})$ represented by $2g$ embedded circles.]

3. Determine the intersection form of M.

Exercise 2. (Resolving transverse intersections) Let M be a smooth 4-manifold and Σ_1, Σ_2 embedded surfaces in M of genus g_1, g_2. Suppose that Σ_1, Σ_2 have precisely one transverse intersection.

1. Show that by resolving the transverse intersection between Σ_1 and Σ_2 (cf. the proof of Lemma 3.5) we get an embedded surface Σ in M of genus $g_1 + g_2$.

2. Prove that in $H_2(M; \mathbb{Z})$

   $$[\Sigma] = [\Sigma_1] + [\Sigma_2]$$

   by arguing that the difference of the classes on both sides is the boundary of a (singular) 3-cycle.

Exercise 3. (Intersection form of connected sums) Let M_1 and M_2 be closed, oriented, connected 4-manifolds with intersection forms Q_{M_1} and Q_{M_2} respectively. Show that the intersection form of their connected sum $M_1 \# M_2$ is given by

$$Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2} \ .$$

(please turn)

Exercise 4. (Embedded surfaces in $S^2 \times S^2$) Let $M = S^2 \times S^2$ and consider the homology classes $a, b \in H_2(M; \mathbb{Z})$ defined by

$$a = \left[S^2 \times \{p\}\right], \quad b = \left[\{q\} \times S^2\right],$$

where $p, q \in S^2$ are arbitrary points.

1. Prove that the class na for every $n \in \mathbb{Z}$ can be represented by an embedded sphere.

2. Prove that the class $na + mb$ for every $n, m \in \mathbb{Z} \setminus \{0\}$ can be represented by an embedded surface Σ of genus

$$g = (|n| - 1)(|m| - 1).$$

You can email the solutions until Tuesday, May 19th at 9:00 am.