LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2020

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Mathematical Gauge Theory II

Sheet 3

Exercise 1. (Associated $\mathrm{Spin}^c(n)$-bundles, cf. Lemma 2.36) Let $Q \longrightarrow X$ be a principal $\mathrm{Spin}^c(n)$-bundle with an isomorphism $Q/S^1 \cong \mathrm{Fr}(H)$ for an oriented Euclidean vector bundle $H \longrightarrow X$. Consider the vector bundle $V \longrightarrow X$ associated to Q by the standard representation

$$\begin{aligned} \mathrm{Spin}^c(n) &\longrightarrow \mathrm{U}(N) \\ (\tau, \sigma) &\mapsto \quad \sigma\ . \end{aligned}$$

Show that the standard Clifford module γ_0 induces a Clifford module γ for V.

Exercise 2. (Twisting of Spin^c-structures I, cf. Lemma 2.20) Let $(V_{\mathfrak{s}}, \gamma_{\mathfrak{s}})$ be a Spin^c-structure and L a line bundle. Show that the pair $(V_{\mathfrak{s}'}, \gamma_{\mathfrak{s}'})$ has a Clifford module structure, where $V_{\mathfrak{s}'} :\, = V_{\mathfrak{s}} \otimes L_\delta$, $i \colon \mathrm{End}(V_{\mathfrak{s}}) \longrightarrow \mathrm{End}(V_{\mathfrak{s}'})$ is an isomorphism, and $\gamma_{\mathfrak{s}'} = i \circ \gamma_{\mathfrak{s}}$.

Exercise 3. (Twisting of Spin^c-structures II) Let $\mathfrak{s}$ be a Spin^c-structure on an oriented Riemannian 4-manifold X and E a complex line bundle over X. Show that for the Spin^c-structure $\mathfrak{s}' = \mathfrak{s} \otimes L$ the characteristic line bundle satisfies

$$L_{\mathfrak{s}'} = L_{\mathfrak{s}} \otimes L^2,$$

where $L^2 = L \otimes L$.

Exercise 4. (Hodge dual and differential of 1-forms) Let (X^n, g) be an oriented Riemannian manifold with Levi-Civita connection ∇ and $\eta \in \Omega^1(X)$ a 1-form. Let $p \in X$ and $e_1, \ldots, e_n$ an oriented local frame for TX on an open neighbourhood around p such that

$$(\nabla e_i)(p) = 0 \quad \forall i \in \{1, \ldots, n\}.$$

Prove that in p

$$* d * \eta = \sum_{i=1}^{n} L_{e_i} \eta(e_i),$$

where $*$ denotes the Hodge star.

You can email the solutions until Tuesday, May 12th at noon.