



Mathematical Gauge Theory II

Sheet 1

Exercise 1. (The standard Clifford module of \mathbb{R}^4) Let e_0, e_1, e_2, e_3 be an orthonormal basis of \mathbb{R}^4 with the standard Euclidean scalar product. Define a linear map $\gamma: \mathbb{R}^4 \rightarrow \text{End}(\mathbb{C}^4)$ by

$$\gamma(e_j) = A_j = \begin{pmatrix} 0 & -B_j^\dagger \\ B_j & 0 \end{pmatrix}$$

where

$$B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Prove that (\mathbb{C}^4, γ) with the standard Hermitian scalar product is a Clifford module for $(\mathbb{R}^4, g_{\text{can}})$.

Exercise 2. (Clifford multiplication with forms) Let (\mathbb{C}^4, γ) be the standard Clifford module for $(\mathbb{R}^4, g_{\text{can}})$ from Exercise 1. Prove that:

1.

$$\gamma(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

2. Under γ the self-dual two-forms

$$\begin{aligned} e_0 \wedge e_1 + e_2 \wedge e_3 \\ e_0 \wedge e_2 - e_1 \wedge e_3 \\ e_0 \wedge e_3 + e_1 \wedge e_2 \end{aligned}$$

act non-trivially on \mathbb{C}_+^2 as $2B_1, 2B_2$ and $2B_3$, respectively, and are zero on \mathbb{C}_-^2 .

3. The map γ induces isomorphisms

$$\begin{aligned} (\Lambda^1(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)) \otimes \mathbb{C} &\cong \text{Hom}(\mathbb{C}_+^2, \mathbb{C}_-^2) \oplus \text{Hom}(\mathbb{C}_-^2, \mathbb{C}_+^2) \\ \Lambda_\pm^2(\mathbb{R}^4) \otimes \mathbb{C} &\cong \text{End}_0(\mathbb{C}_\pm^2) \\ \Lambda^4(\mathbb{R}^4) \otimes \mathbb{C} &\cong \mathbb{C} \cdot \text{Id}_{\mathbb{C}_\pm^2}, \end{aligned}$$

where End_0 denotes the trace-free endomorphisms.

(please turn)

Exercise 3. (Schur's Lemma) Let (V, γ) be an irreducible Clifford module. Prove that every automorphism of (V, γ) , i.e. every isomorphism $f: V \rightarrow V$ of Clifford modules, is of the form $f(\phi) = \lambda\phi$ for some constant $\lambda \in S^1$.

Exercise 4. (Anti-linear automorphisms I) Let $J: V \rightarrow V$ be a complex anti-linear automorphism of a standard (i.e. irreducible) Clifford module. Show that $J^2 = \pm \text{Id}_V$.

You can email the solutions until Tuesday, April 28th at noon.