LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
MÜNCHEN

MATHEMATISCHES INSTITUT

Summer term 2019

Prof. D. Kotschick
G. Placini

Topology II

Sheet 5

Exercise 1.

Let $T^n = S^1 \times \cdots \times S^1$ be the n-dimensional torus and R a commutative ring.

a) Compute the cohomology modules $H^i(T^n; R)$ for all $i \in \mathbb{Z}$.

b) Determine the cup product structure on $H^*(T^n; R)$.

Exercise 2.

a) Show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $f^* \colon H^i(S^k \times S^l; R) \longrightarrow H^i(S^{k+l}; R)$, assuming $k, l, i > 0$.

b) Show there is no map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ inducing a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m; \mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2)$ if $n > m$.
What is the corresponding result for maps $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$?

Exercise 3.

a) Let $A, B \subset X$ be open sets in a space X and R a commutative ring. Prove that the cup product $H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R)$ is well defined.

b) Show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. Generalize to the situation that X is the union of n contractible open subsets, to show that all cup products $\alpha_1 \smile \cdots \smile \alpha_n$ of n positive-dimensional classes are zero.

Exercise 4. Prove that $S^1 \times S^2$ and $S^1 \vee S^2 \vee S^3$ have the same cohomology groups for any coefficients but not the same cup product structure. Conclude that the two spaces are not homotopy equivalent.
[Hint: Recall that $H^1(I, \partial I) \cong H^1(S^1)$ and $H^{k+1}(Y \times I, Y \times \partial I) \cong H^{k+1}(Y \times S^1)$ for all spaces Y.]

Hand in: during the exercise class on Monday, June 3rd.