Summer term 2018

Prof. T. Vogel
G. Placini

Topology II

Sheet 8

Exercise 1. The **direct limit** $\varinjlim G_i$ of a sequence of homomorphisms of abelian groups $G_1 \xrightarrow{\alpha_1} G_2 \xrightarrow{\alpha_2} G_3 \to \cdots$ is defined to be the quotient of the direct sum $\oplus_i G_i$ by the subgroup consisting of elements of the form $(g_1, g_2 - \alpha_1(g_1), g_3 - \alpha_2(g_2), \ldots)$.

a) Prove that every element of $\varinjlim G_i$ is represented by an element $g_i \in G_i$ for some i, and two such representatives $g_i \in G_i$ and $g_j \in G_j$ define the same element of $\varinjlim G_i$ if and only if they have the same image in some G_k under the appropriate composition of α_l's.

b) Give a description of $\varinjlim G_i$ where $G_i = \mathbb{Z}$ for all i and all maps are multiplication by a prime p.

c) Show that $\varinjlim G_i = \mathbb{Q}$ when $G_i = \mathbb{Z}$ and the map α_i is multiplication by i for all i.

Exercise 2. Let X be a topological space and $C_1 \subset C_2 \subset C_3 \subset \cdots$ be subsets of X such that any compact set $K \subset X$ is contained in C_i for some i. Prove that $\varinjlim H_n(C_i; G) = H_n(X; G)$.

Exercise 3. Given a sequence of group homomorphisms $\cdots \to G_2 \xrightarrow{\alpha_2} G_1 \xrightarrow{\alpha_1} G_0$, the **inverse limit** $\varprojlim G_i$ is defined to be the subgroup of $\Pi_i G_i$ consisting of sequences (g_i) with $\alpha_i(g_i) = g_{i-1}$ for all i.

a) Show that $\mathrm{Hom}(\varinjlim G_i, G) = \varprojlim \mathrm{Hom}(G_i, G)$ for any abelian group G.

b) Given X and C_i's as in exercise 2, define a natural map $\lambda : H^n(X; G) \to \varprojlim H^n(C_i; G)$.

c) Prove that the map λ is an isomorphism when $G = \mathbb{Q}$.

Exercise 4.

a) Given an abelian group G and a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ show that there exists an exact sequence

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to \mathrm{Ext}(G, A) \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0$$

(please turn)

b) Prove that the group $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ is uncountable.

[Hint: Construct a homomorphism for each sequence $a_1, a_2, \ldots \in \mathbb{Q}/\mathbb{Z}$ with $na_n = a_{n-1}$.]

c) Use the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ to prove that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is uncountable.

Exercise 5. The **mapping telescope** of a sequence of maps $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots$ is the union of the mapping cylinders M_{f_i} with the copies of X_i in M_{f_i} and $M_{f_{i-1}}$ identified for all i.

a) Consider the mapping telescope T of the sequence $S^1 \xrightarrow{f_1} S^1 \xrightarrow{f_2} S^1 \to \cdots$ with $\deg(f_i) = i$. Use exercise 2 to compute $H_*(T, \mathbb{Z})$.

b) Use the Universal Coefficient Theorem and exercise 4 to conclude that the analogue of exercise 2 in cohomology (i.e. a generalization to arbitrary abelian groups of 3.c)) cannot hold.

Hand in: during the lecture on Monday, June 11th.