Summer term 2018

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Topology II

Sheet 4

Exercise 1. Let X, Y be topological spaces and G, H, L abelian groups.

a) Show that a group homomorphism $\varphi : G \to H$ induces a chain map $\varphi_* : C_*(X; G) \to C_*(X; H)$.

b) Show that a short exact sequence $0 \to G \to H \to L \to 0$ induces a long exact sequence in homology:

$$\cdots \longrightarrow H_i(X; L) \xrightarrow{B} H_{i-1}(X; G) \to H_{i-1}(X; H) \to H_{i-1}(X; L) \xrightarrow{B} H_{i-2}(X; G) \longrightarrow \cdots$$

c) Prove the naturality of the connecting homomorphism $B : H_i(-; \mathbb{Z}_2) \to H_{i-1}(-; \mathbb{Z}_2)$ associated to the sequence $0 \to \mathbb{Z}_2 \xrightarrow{\cdot 2} \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ i.e., show that $f_* \circ B = B \circ f_*$ for a continuous map $f : X \to Y$.

Exercise 2. Give a description of the map $B : H_i(X; \mathbb{Z}_2) \to H_{i-1}(X; \mathbb{Z}_2)$ from the previous exercise when $X = \mathbb{R}\mathrm{P}^\infty$.

Exercise 3. Prove the equality

$$\sum_i (-1)^i \cdot \mathrm{trace}\left(f_i : C_i^{CW}(|K|; \mathbb{R}) \longrightarrow C_i^{CW}(|K|; \mathbb{R})\right)$$
$$= \sum_i (-1)^i \cdot \mathrm{trace}\left(f_i : H_i(|K|; \mathbb{R}) \longrightarrow H_i(|K|; \mathbb{R})\right)$$

where $f : |K| \to |K|$ is a continuous map and K a simplicial complex.

Exercise 4. Prove the following statements:

a) Every continuous map $\mathbb{R}\mathrm{P}^n \longrightarrow \mathbb{R}\mathrm{P}^n$ for n even has a fixed point.

b) If $f : |K| \longrightarrow |K|$ is homotopic to the identity and $\chi(|K|) \neq 0$, then f has a fixed point.

Hand in: during the lecture on Monday, May 7th.