Summer term 2018

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Geometric Group Theory

Sheet 9

Exercise 1.

a) Is there a finite generating set $S \subset \mathbb{Z}^2$ satisfying

$$\beta_{\mathbb{Z}^2,S}(42) = 2018 \ \ ?$$

b) Let $n \in \mathbb{N}$. Is there a finite generating set $S \subset \mathbb{Z}^n$ such that for all $r \in \mathbb{N}$ we have

$$\beta_{\mathbb{Z}^n,S}(r) = \{-r, \dots, r\}^n \ \ ?$$

Exercise 2. Let $n \in \mathbb{N}$. Show that a growth function of $\mathbb{Z}^n$ is equivalent to $(x \mapsto x^n)$.

Exercise 3.

a) Show that the functions $(x \mapsto a^x)$ and $(x \mapsto a'^x)$ are equivalent for all $a, a' \in \mathbb{R}_{>1}$.

b) Let $a \in \mathbb{R}_{>1}$ and $a' \in \mathbb{R}_{>0}$. Show that $(x \mapsto a^x)$ dominates $(x \mapsto x^{a'})$.

c) Let $a \in \mathbb{R}_{>1}$ and $a' \in \mathbb{R}_{>0}$. Show that $(x \mapsto x^{a'})$ does not dominate $(x \mapsto a^x)$.

d) Find a generalised growth function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $f \prec (x \mapsto e^x)$, $f \not\sim (x \mapsto e^x)$, and $(x \mapsto x^a) \prec f$ for all $a \in \mathbb{R}_{>0}$.

Exercise 4. Let G be a finitely generated group that acts on a set X. Moreover, let $a, b \in G$ and let $A, B \subset X$ be non-empty subsets with the following properties:

$$A \cap B = \emptyset, \qquad a \cdot (A \cup B) \subset B, \qquad b \cdot (A \cup B) \subset A.$$

Show that then G has exponential growth.

You can hand in your solutions during the exercise classes.