Summer term 2018

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Geometric Group Theory

Sheet 5

Exercise 1.

a) Sketch a spanning tree for the action of the group $\mathbb{Z}$ on the Cayley graph $\mathrm{Cay}(F(\{a,b\}),\{a,b\})$, where the action is given by left translation by the powers of a. Are all spanning trees of this action isomorphic?

b) The group $\mathbb{Z}/4$ acts on $\mathrm{Cay}(\mathbb{Z}^2,\{(1,0),(0,1)\})$ via rotation by π; i.e., the generator $[1]\in\mathbb{Z}/4$ acts by rotation by π around 0. Sketch a spanning tree for this action. Are all spanning trees of this action isomorphic?

Exercise 2. The rank $\mathrm{rk}G$ of a group G is the minimal cardinality of a generating set of G. The **rank gradient** $\mathrm{rg}G$ of a finitely generated group G is defined by

$$\mathrm{rg}G := \inf_{H\in S(G)} \frac{\mathrm{rk}H}{[G:H]};$$

where $S(G)$ denotes the set of all finite index subgroups of G.

a) Determine $\mathrm{rg}Z^d$ for all $d\in\mathbb{N}$.

b) Determine the rank gradient of finitely generated free groups.

Exercise 3. We consider the homeomorphism

$$\begin{aligned} f : [0,1] &\longrightarrow [0,1] \\ t &\mapsto \begin{cases} 4\cdot t & \text{if } t\in[0,\frac{1}{5}] \\ \frac{4}{5}+\frac{1}{4}\cdot\left(t-\frac{1}{5}\right) & \text{if } t\in[\frac{1}{5},1] \end{cases} \end{aligned}$$

and the maps

$$\begin{aligned} a : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\mapsto \lfloor t\rfloor + f\left(\{t\}\right) \end{aligned}$$

(where $\lfloor\cdot\rfloor$ denotes the lower integral part and $\{\cdot\} := \mathrm{Id}-\lfloor\cdot\rfloor$ denotes the fractional part) and

$$b := c\cdot a\cdot c^{-1},$$

where $c : t\mapsto t-\frac{1}{2}$.

(please turn)

a) Show that a and b are self-homeomorphisms of $\mathbb{R}$.

b) Show that a and b generate a free group of rank 2 in the self-homeomorphism group of $\mathbb{R}$.

Exercise 4. Let $n \in \mathbb{N}$ with $n \geq 3$. Show that $\mathrm{SL}(n, \mathbb{Z})$ does not contain a free group of finite index.
[Hint: Try to find a subgroup of $\mathrm{SL}(n, \mathbb{Z})$ that is isomorphic to $\mathbb{Z}^2$.]

You can hand in your solutions during the exercise classes.