LUDWIG-
MAXIMILIANS-
UNIVERSITÄT
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MATHEMATISCHES INSTITUT

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Geometric Group Theory

Sheet 2

Exercise 1. Let $\alpha_1 : A \to G_1$ and $\alpha_2 : A \to G_2$ be injective group homomorphisms. Consider the amalgamated product $G = G_1 *_A G_2$ as the pushout of G_1 and G_2 over A with respect to α_1 and α_2:

$$\begin{tikzcd} A \arrow[r, "\alpha_2"] \arrow[d, "\alpha_1"'] & G_2 \arrow[d, "\beta_2"] \\ G_1 \arrow[r, "\beta_1"'] & G \end{tikzcd}$$

Show that the homomorphism $\beta_1 \circ \alpha_1 = \beta_2 \circ \alpha_2$ is injective.

Exercise 2.

a) Is the additive group $\mathbb{Q}$ finitely generated?

b) Is the symmetric group S_X of an infinite set X finitely generated?

Exercise 3. Let S be a set and let F be the free group generated by S.

a) Prove that if $S' \subset F$ is a generating set of F, then $|S'| \geq |S|$.

b) Conclude that all free generating sets of a free group have the same cardinality.

c) Show that the free group generated by two elements contains a subgroup that cannot be generated by two elements.

Exercise 4. For $n \in \mathbb{N}$ we define

$$G_n := \left\langle a_1, \ldots, a_n, b_1, \ldots, b_n \middle| \prod_{i=1}^{n} [a_i, b_i] \right\rangle.$$

a) Prove that for all $n, m \in \mathbb{N}$ we have $G_n \cong G_m$ if and only if $n = m$.

b) For which $n \in \mathbb{N}$ is the group G_n Abelian?

You can hand in your solutions during the exercise classes.