Summer term 2017

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Topology II

Sheet 5

Exercise 1. Using the cup product structure, show there is no map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^m$ inducing a nontrivial map $H^1(\mathbb{R}\mathrm{P}^m;\mathbb{Z}_2) \to H^1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2)$ if $n > m$. What is the corresponding result for maps $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$?

Exercise 2 (Borsuk-Ulam Theorem)**.** Prove by contradiction that for every map $f : S^n \to \mathbb{R}^n$ there exists an $x \in S^n$ such that $f(x) = f(-x)$. Namely, suppose $f : S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ for all $x \in S^n$ and define $g : S^n \to S^{n-1}$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

so $g(-x) = -g(x)$ and g induces a map $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^{n-1}$. Conclude by using the previous exercise.

Exercise 3. Show that the ring $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_{2k})$ is isomorphic to $\mathbb{Z}_{2k}[\alpha,\beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$ where $\deg(\alpha) = 1$ and $\deg(\beta) = 2$.
[Hint: Use the coefficient map $\mathbb{Z}_{2k} \to \mathbb{Z}_2$ and the computation of the ring $H^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}_2)$.]

Exercise 4. Using cup products, show that every map $S^{k+l} \to S^k \times S^l$ induces the trivial homomorphism $f^* : \tilde{H}^*(S^k \times S^l; R) \to \tilde{H}^*(S^{k+l}; R)$, assuming $k, l > 0$.

Hand in: Tuesday, June 13th, during the exercise class.