Summer term 2017

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Topology II

Sheet 4

Exercise 1. Compute the cup product structure in $H^*(\Sigma_g;\mathbb{Q})$ for Σ_g the closed orientable surface of genus g by using the quotient map from Σ_g to the one point union of g tori.

Exercise 2. For Σ_g the closed orientable surface of genus $g \geq 1$, show that for each nonzero $\alpha \in H^1(\Sigma_g;\mathbb{Q})$ there exists $\beta \in H^1(\Sigma_g;\mathbb{Q})$ with $\alpha \smile \beta \neq 0$. Deduce that Σ_g is not homotopy equivalent to a one point union $X \vee Y$ of CW complexes with nontrivial reduced homology.

Exercise 3. Using the cup product $H^k(X,A;R) \times H^l(X,B;R) \to H^{k+l}(X,A\cup B;R)$, show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X;R)$ are zero. Generalize to the situation that X is the union of n contractible open subsets, to show that all cup products $\alpha_1 \smile \cdots \smile \alpha_n$ of n positive-dimensional classes are zero.

Exercise 4. Prove that $S^1\times S^2$ and $S^1\vee S^2\vee S^3$ have the same cohomology groups for any coefficients but not the same cup product structure. Conclude that the two spaces are not homotopy equivalent. [Hint: Recall that $H^1(I,\partial I) \simeq H^1(S^1)$ and $H^{k+1}(Y\times I, Y\times \partial I) \simeq H^{k+1}(Y\times S^1)$ for all spaces Y.]

Hand in: Tuesday, May 23rd, during the exercise class.