

Flux-Across-Surfaces Theorem for a Dirac Particle

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October 2002

Abstract

We consider the asymptotic evolution of a relativistic spin- $\frac{1}{2}$ particle. i.e. a particle whose wavefunction satisfies the Dirac equation with external static potential. We prove that the probability for the particle crossing a (detector) surface converges to the probability, that the direction of the momentum of the particle lies within the solid angle defined by the (detector) surface, as the distance of the surface goes to infinity. This generalizes earlier non relativistic results, known as flux across surfaces theorems, to the relativistic regime.

1 Introduction

In scattering experiments the scattered particles are measured at a macroscopic distance, but the computations of scattering cross sections are based on the distribution of the wavefunction in momentum space. Therefore a relationship between the crossing probability through a far distant detector surface and the shape of the wavefunction in momentum space is needed.

This relationship is given by the flux-across-surfaces theorem, which - as a problem in mathematical physics - has been formulated by Combes, Newton and Shtokhamer [3], see also [5, 8]. For scattering states (material on scattering states for the Dirac equation is in [14]) the theorem asserts that the probability of crossing a far distant surface (physical interaction with the detector is neglected) subtended by a solid angle is equal to the probability that the

scattered particle will, in the distant future, have a momentum, whose direction lies in that same solid angle. Moreover, the probability, that the particle will cross the detector within a certain area is given by the integral of the flux over that area and time. This has been proven for Schrödinger evolutions in great generality, see for instance [15, 2, 1, 16, 12, 6].

We consider here wavefunctions $\psi_t \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ which satisfy the Dirac-equation (conveniently setting $c = \hbar = 1$)

$$i \frac{\partial \psi_t}{\partial t} = -i \sum_{l=1}^3 \alpha_l \partial_l \psi_t + \mathcal{A} \psi_t + \beta m \psi_t \equiv H \psi_t \quad (1)$$

where

$$\alpha_l = \begin{pmatrix} 0 & \sigma_l \\ \sigma_l & 0 \end{pmatrix}; \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; l = 1, 2, 3 \quad (2)$$

σ_l being the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\mathbf{1}$ the 2×2 -unit matrix and \mathcal{A} the 4-potential in the form

$$\mathcal{A} := A_0 + \mathbf{A} \cdot \boldsymbol{\alpha}$$

with $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \alpha_3)$. In the following we will always denote solutions of the Dirac equation by ψ_t and by ψ_0 the "time zero" wavefunction.

\mathcal{A} is an external static four-potential, which satisfies condition A (see 3), which concerns smoothness and is for the sake of simplicity taken stronger than needed:

$$\textbf{Condition A:} \quad \mathcal{A}(\mathbf{x}) \in C^\infty \quad \exists M, \xi > 0: \quad |\mathcal{A}(\mathbf{x})| \leq M \langle x \rangle^{-(4+\xi)}. \quad (3)$$

where $\langle x \rangle = |x| + 1$ and the norm $|\cdot|$ is defined as:

$$|B| := \sup_{\|\varphi\|_s=1} \|B\varphi\|_s$$

where

$$\|\varphi\|_s := \langle \varphi, \varphi \rangle^{\frac{1}{2}}$$

with the inner product in spin space

$$\langle \cdot, \cdot \rangle : \mathbb{C} \|\varphi\|_s(\mathbf{x})^4 \otimes \mathbb{C}^4 \rightarrow \mathbb{C} \quad \langle \varphi, \chi \rangle := \sum_{l=1}^4 \overline{\varphi_l} \chi_l .$$

Often we have spinors depending on \mathbf{x} , in that case we have $\|\varphi\|_s(\mathbf{x})$.

The continuity equation involving the quantum flux of a relativistic spin $\frac{1}{2}$ particle reads

$$\frac{\partial}{\partial t} \overline{\psi_t} \psi_t = \nabla \cdot \underline{\mathbf{j}}, \quad (4)$$

whereas the 4-flux is defined for any $\varphi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ by

$$\underline{\mathbf{j}} = \begin{pmatrix} j_0 \\ \mathbf{j} \end{pmatrix} = \langle \varphi, \underline{\alpha} \varphi \rangle, \quad (5)$$

$$\text{with } \underline{\alpha} = \begin{pmatrix} 1 \\ \boldsymbol{\alpha} \end{pmatrix} .$$

For notational convenience we sometimes omit the dependence on \mathbf{x} . Furthermore we have the usual L^2 -Norm on the space of 4-spinors given by

$$\|\varphi\| = \left(\int \|\varphi\|_s^2 d^3x \right)^{\frac{1}{2}} .$$

We introduce the Fourier transform of $\varphi(\mathbf{x})$ as representation in the generalized basis (13) of the free Hamiltonian ,i.e.

$$\widehat{\varphi}_s(\mathbf{k}) = \int (2\pi)^{-\frac{3}{2}} \langle \varphi_{\mathbf{k}}^s(\mathbf{x}), \varphi(\mathbf{x}) \rangle d^3x \quad \widehat{\varphi}(\mathbf{k}) := \sum_{s=1}^2 s_{\mathbf{k}}^s \widehat{\varphi}_s(\mathbf{k}) . \quad (6)$$

We denote by x the euclidian length of \mathbf{x} .

We assume that asymptotic completeness holds, i.e. that the wave operators exist on the spectral subspace \mathcal{H}_{ac} of the continuous positive spectrum ("scattering state") of the Dirac-Hamiltonian: Let ψ_{out} denote the wavefunction of the free asymptotic of a scattering state ψ then

$$\lim_{t \rightarrow \infty} \| e^{-iH_0 t} \psi_{\text{out}} - e^{-iH t} \psi \| = 0 .$$

ψ_{out} is given by the wave operator:

$$\Omega_+ = \lim_{t \rightarrow \infty} e^{iH t} e^{-iH_0 t} \quad \psi = \Omega_+ \psi_{\text{out}} .$$

The existence of the wave operators and asymptotic completeness has been proven for short range potentials. See e.g. Thaller [14].

We remark (see Lemma 3.4(d))(41), that the Fourier transform $\widehat{\psi}_{\text{out},s}(\mathbf{k})$ of ψ_{out} equals the generalized Fourier transform ψ_s^\ddagger of ψ in the generalized eigen-basis of the Dirac hamiltonian with potential.

In general, we do not have much information about scattering states. One can prove the flux across surface theorem with conditions merely on the "out"-states, where the corresponding properties of the scattering states are hidden in the mapping properties of the wave operators, or, better, in the smoothness properties of the generalized eigenfunctions. On the other hand, one would like to be sure, that such conditions are not too restrictive on the set of scattering states.

We introduce the set \mathcal{G} of functions $\widehat{\psi}_{\text{out}}$, for which the flux across surfaces can naturally be proven:

$$f(\mathbf{k}) \in \mathcal{G} \iff \begin{cases} \exists M \in \mathbb{R} : & \| \partial_k^j f(\mathbf{k}) \|_s \leq M \langle k \rangle^{-n} & \text{for } j = 0, 1, 2; n \in \mathbb{N} \\ \forall \mathbf{k} \neq 0 : & \| k^{|\gamma|-1} D_{\mathbf{k}}^\gamma f(\mathbf{k}) \|_s \leq M \langle k \rangle^{-n} & \text{for } n \in \mathbb{N} \end{cases} \quad (7)$$

where $\gamma = (\gamma_1; \gamma_2; \gamma_3)$ is a multi-index with $|\gamma| \leq 2$, $D_{\mathbf{k}}^\gamma := \partial_{k_1}^{\gamma_1} \partial_{k_2}^{\gamma_2} \partial_{k_3}^{\gamma_3}$ and ∂_k is the partial derivative with respect to the radial coordinate k .

This set maps under the wave operator to a dense set in the set of scattering states. After the theorem we shall give under more restrictive conditions more detailed information on the set of scattering states for which the theorem holds.

The paper is organized as follows: In the next section we shall state the theorem. We shall also give its formulation in covariant form, but we shall prove the theorem using the rest frame of the detector and the potential.

The following sections contain the proof of the theorem. We first prove the statement for the free case ($A = 0$) and then for the case of nonzero potential. Both are done in section 3. The proof relies almost entirely on the stationary phase method, which we need to adapt to our purposes. The main lemma is lemma (3.1), whose lengthy technical proof is put in the Appendix 4.1.

The difficulty we have to face and which makes this paper not a simple generalization of the results in the Schrödinger situation is, that the time evolution with the Dirac hamiltonian is not of a "nice" form for the stationary phase method to be easily applied to. The Schrödinger case is easier. On the other hand, the expression for the flux needs no differentiability of the wavefunction and one might be lead to believe, that to describe scattering in the relativistic regime is simpler—in particular less restrictive theorems should result. One may even get the idea, that asymptotic completeness and the flux across surfaces theorem become more or less equivalent statements in the relativistic regime. But we are far from that. Nevertheless, that we require smoothness and good decay on the potential may well be due to our method of proof.

We also need information about the generalized eigenfunctions of the Dirac hamiltonian with external potential, see Lemma 3.4, whose proof is also put into the Appendix 4.3. The appendix, which in fact is almost half of the paper, contains other tedious technical details.

Acknowledgement: Very helpful discussions with Stefan Teufel are gratefully acknowledged. We would also like to thank the referee for a very critical reading of the manuscript, which let to a clear improvement of the paper.

2 The Theorem

The flux-across-surfaces theorem deals with the flux \mathbf{j} integrated over a spherical surface at a far distance and asserts that

1. the absolute value of the flux and the flux itself yield the same asymptotics, allowing to interpret the flux integral as crossing probability [5, 4],
2. the crossing probability equals the probability for the momentum to lie within the cone defined by the surface.

Theorem 2.1 *Let ψ be a scattering state with outgoing free asymptotic $\widehat{\psi}_{out}$, whose Fourier transform $\widehat{\psi}_{out}$ lies in \mathcal{G} (cf. γ). Let $R^2 d\Omega$ be the surface element at distance R with solid angle differential $d\Omega$ and let \mathbf{n} denote the outward normal of the surface element. Furthermore let S be a subset of the unit sphere. Then for all $t_i \in \mathbb{R}$:*

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_S \int_{t_i}^{\infty} j(\mathbf{R}, t) dt R^2 d\Omega &= \lim_{R \rightarrow \infty} \int_S \int_{t_i}^{\infty} \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega \\ &= \int_S \int_0^{\infty} \langle \widehat{\psi}_{out}(\mathbf{k}), \widehat{\psi}_{out}(\mathbf{k}) \rangle k^2 dk d\Omega \quad , \end{aligned} \quad (8)$$

Observing that $\| \widehat{\psi}_{out} \|(\mathbf{k})$ does not depend on time, we can choose a coordinate system $t' = t - t_i$, so that we may for definiteness always put $t_i = 0$ in (8).

The conditions on ψ_{out} can be translated into more detailed conditions on the scattering states under more restrictive conditions on the potential: Let

Condition B $|\partial_x^n A(\mathbf{x})| \in L^2(\mathbb{R}^3) \forall n \in \{0, 1, 2, \dots\} \quad \exists M |A(\mathbf{x})| \leq M \langle x \rangle^{-6} .$

Then (for the proof see Appendix 4.4)

Lemma 2.2

$$\widehat{\psi}_{out}(\mathbf{k}) \in \mathcal{G} \Leftrightarrow \psi(\mathbf{x}) \in \widehat{\mathcal{G}} \quad (9)$$

where $\widehat{\mathcal{G}}$ is the space of functions $\psi(\mathbf{x}) \in \mathcal{H}_{ac}$ with $x^j \nabla^n \psi(\mathbf{x}) \in L^2$ for all $j = 0, 1, 2; n \in \mathbb{N}_0$, where $\nabla := -i \sum_{l=1}^3 \alpha_l \partial_l$.

2.1 Covariant form of the theorem

As we deal with a relativistic regime, it might be of interest to have also a covariant formulation of the theorem. As $\langle \widehat{\psi}_{out}, \widehat{\psi}_{out} \rangle$ is not conserved under Lorentz function we use

$$\widehat{\psi}_{out}^{LI}(\underline{k}) = (k^2 + m^2)^{\frac{1}{4}} \widehat{\psi}_{out}(\underline{k}),$$

of which it is known, that $\langle \widehat{\psi}_{out}^{LI}, \widehat{\psi}_{out}^{LI} \rangle$ is a Lorentz-scalar (see for instance [9]). Then the flux-across-surfaces theorem reads in a general and covariant way:

Theorem 2.3 *Let the conditions of Theorem 2.1 be satisfied. Let*

$$\underline{x} \diamond \underline{y} := x_0 y_0 - \sum_{j=1}^3 x_j y_j$$

be the Minkowski scalar product. Then for any subspace $Z \subseteq \{\underline{x} \mid \underline{x} \diamond \underline{x} = m^2\} \subset \mathbb{R}^4$ and any smooth scalar function $\eta(\underline{x})$, nonequal to zero for all \mathbf{x} :

$$\lim_{\lambda \rightarrow \infty} \int_{\widetilde{Z}(\lambda)} \underline{j}(\mathbf{x}) \diamond \underline{n} d\widetilde{\sigma} = \int_Z \langle \widehat{\psi}_{out}^{LI}(\underline{k}), \widehat{\psi}_{out}^{LI}(\underline{k}) \rangle d\sigma. \quad (10)$$

where

$$\widetilde{Z}(\lambda) := \{\underline{y} \mid \exists \underline{x} \in Z : \underline{y} = \lambda \eta(\underline{x}) \underline{x}\} \subset \mathbb{R}^4$$

and $d\sigma$ is the invariant measure on Z , $d\widetilde{\sigma}$ the invariant measure on \widetilde{Z} and \underline{n} is the vector orthogonal on \widetilde{Z} with Lorentz length one.

This formulation may perhaps not be directly guessed, but once one understands its basics like (19), this formulation becomes clear: The arbitrariness of the scalar function η follows directly from (19), observing that

$$\lim_{\lambda \rightarrow \infty} \psi(\lambda \underline{k}) = \lim_{\lambda \rightarrow \infty} \psi(\lambda \eta(\underline{k}) \underline{k}).$$

Physically this is related to the fact, that (on big scales) it is possible to "catch" any part of the wave-function in different ways (for example by using a detector which is "close" and catches the wavefunction at an "early" time or one uses a far detector at a later time-interval).

Let us explain how (8) follows from (10). We choose a set Z whose projection on the $t = 0$ -subspace is a cone with angular distribution S :

$$Z = \{ \underline{k} \mid \frac{\mathbf{k}}{k} \in S \} \cap \{ \underline{k} \diamond \underline{k} = m^2 \} .$$

The invariant measure on the mass hyperboloid $d\sigma = \frac{d^3 k}{\sqrt{k^2 + m^2}}$ we get for the right hand side of (10)

$$\int_Z \langle \widehat{\psi}_{\text{out}}^{LI}(\underline{k}), \widehat{\psi}_{\text{out}}^{LI}(\underline{k}) \rangle d\sigma = \int_S \int_0^\infty \langle \widehat{\psi}_{\text{out}}(\mathbf{k}), \widehat{\psi}_{\text{out}}(\mathbf{k}) \rangle k^2 dk d\Omega . \quad (11)$$

For the left hand side of (10) we take:

$$\eta(\underline{x}) := \frac{1}{x} \quad x \neq 0 .$$

As both integrands in (10) are bounded, a small neighborhood of $\mathbf{x} = 0$ can be neglected. For constant λ , \widetilde{Z} represents a radial surface with arbitrary time $t \geq 0$. So we have:

$$\lim_{\lambda \rightarrow \infty} \int_{\widetilde{Z}(\lambda)} \underline{j}(\mathbf{x}) \diamond \underline{n} d\widetilde{\sigma} = \lim_{R \rightarrow \infty} \int_S \int_{t_i}^\infty j(\mathbf{R}, t) dt R^2 d\Omega . \quad (12)$$

3 The proof

3.1 Scattering into cones heuristics

The flux-across-surfaces theorem is based on an asymptotic connection between the shape of the wavefunction in momentum space and in ordinary space. This is often referred to as the scattering into cones theorem, which has been proven for non-relativistic particles by Dollard [7]. For that one chooses a certain parameterization of \mathbb{R}^4 and evaluates the wavefunction, as the parameter of the parameterizations goes to infinity. In the non-relativistic case, it is easiest to choose time as the parameter of the parameterization. In the relativistic case it is simplest to have lorentz-invariant three-dimensional subspaces of the time-like part of \mathbb{R}^4 as leaves of

the parametrization¹. This can easily be done, by choosing a lorentz-vector as argument of ψ , i.e. a vector \underline{x} with $\underline{x} \diamond \underline{x} = x_0^2 - \mathbf{x} \cdot \mathbf{x} = \lambda m^2$. Set $\psi(\lambda \underline{k}) = \psi(\mathbf{x} = \lambda \mathbf{k}, t = \lambda \sqrt{k^2 + m^2})$. We denote the two different eigenstates of momentum \mathbf{k} of the free Hamiltonian with positive energy by $\varphi_{\mathbf{k}}^s$, whereas the s labels the two different spins our electron may have. In the standard representation these eigenstates can be written as:

$$\varphi_{\mathbf{k}}^s = e^{i\mathbf{k} \cdot \mathbf{x}} s_{\mathbf{k}}^s, \quad (13)$$

where the $s_{\mathbf{k}}^s$ are:

$$s_{\mathbf{k}}^1 = (2E_k \widehat{E}_k)^{-\frac{1}{2}} \begin{pmatrix} \widehat{E}_k \\ 0 \\ k_1 \\ k^+ \end{pmatrix} \quad s_{\mathbf{k}}^2 = (2E_k \widehat{E}_k)^{-\frac{1}{2}} \begin{pmatrix} 0 \\ \widehat{E}_k \\ k^- \\ -k_1 \end{pmatrix},$$

where

$$k^\pm = k_2 \pm ik_3 \quad \widehat{E}_k = E_k + m \quad E_k = \sqrt{k^2 + m^2}.$$

(For a detailed calculation of these spinors see ([14]))

The asymptotics result from a stationary phase analysis:

$$\begin{aligned} \psi(\lambda \underline{k}) &= U(t = \lambda \sqrt{k^2 + m^2}) \psi(\lambda \mathbf{k}, 0) \\ &= \sum_{s=1}^2 e^{-iH\lambda\sqrt{k^2+m^2}} \int (2\pi)^{-\frac{3}{2}} \varphi_{\mathbf{k}'}^s(\lambda \mathbf{k}) \widehat{\psi}_s(\mathbf{k}') d^3 k' \\ &= \sum_{s=1}^2 e^{-iH\lambda\sqrt{k^2+m^2}} \int (2\pi)^{-\frac{3}{2}} e^{i\mathbf{k}' \cdot \lambda \mathbf{k}} s_{\mathbf{k}'}^s \widehat{\psi}_s(\mathbf{k}') d^3 k'. \end{aligned}$$

For convenience we define:

¹We only parameterize the time-like region, as for big time-scales the main part of our wavefunction will be in this region.

$$\widehat{\psi}(\mathbf{k}') = \sum_{s=1}^2 s_{\mathbf{k}'}^s \widehat{\psi}_s(\mathbf{k}').$$

This leads to:

$$\begin{aligned} \psi(\lambda \underline{k}) &= e^{-iH\lambda\sqrt{k^2+m^2}} \int (2\pi)^{-\frac{3}{2}} e^{i\mathbf{k}' \cdot \lambda \mathbf{k}} \widehat{\psi}(\mathbf{k}') d^3 k' \\ &= \int (2\pi)^{-\frac{3}{2}} e^{-i\lambda(\sqrt{k'^2+m^2}\sqrt{k^2+m^2}-\mathbf{k}' \cdot \mathbf{k})} \widehat{\psi}(\mathbf{k}') d^3 k'. \end{aligned} \quad (14)$$

In view of the stationary phase method, in the limit $\lambda \rightarrow \infty$ only a small neighborhood of the stationary point of the phase function

$$h(\mathbf{k}') := (\sqrt{k'^2+m^2}\sqrt{k^2+m^2}-\mathbf{k}' \cdot \mathbf{k})$$

will be relevant for the integral. The stationary point is given by:

$$\nabla_{\mathbf{k}'} h(\mathbf{k}_{stat}) = 0 \Rightarrow \mathbf{k}_{stat} = \mathbf{k} \quad (15)$$

Without loss of generality we can set $k_2 = k_3 = 0$. Near the stationary point the phase is to second order:

$$-i\lambda(\sqrt{k'^2+m^2}\sqrt{k^2+m^2}-\mathbf{k}' \cdot \mathbf{k}) \approx -i\lambda(m^2 + \frac{m^2}{2(k^2+m^2)}(k'_1-k)^2 + \frac{1}{2}(k_2'^2+k_3'^2))$$

This in equation(14) leads to

$$\psi(\lambda \underline{k}) \approx \int (2\pi)^{-\frac{3}{2}} e^{-i\lambda(m^2 + \frac{m^2}{2(k^2+m^2)}(k'_1-k)^2 + \frac{1}{2}(k_2'^2+k_3'^2))} \widehat{\psi}(\mathbf{k}') d^3 k',$$

and replacing $\widehat{\psi}(\mathbf{k}')$ by $\widehat{\psi}(\mathbf{k})$ we obtain by integrating the gaussian

$$\psi(\lambda \mathbf{k}) \approx \frac{e^{-i\lambda m^2}}{(i\lambda)^{\frac{3}{2}}} \widehat{\psi}(\mathbf{k}) \sqrt{\frac{k^2}{m^2} + 1}$$

We shall state now the stationary phase result in a some what more general setting, to cover also applications to the potential case considered later:

3.2 The stationary phase

Lemma 3.1 *Let $\tilde{\chi}$ be in \mathcal{G} (see(7)) and let the “phase function” g be*

$$g(\mathbf{k}') = \sqrt{k'^2 + m^2} + a |k'| - \mathbf{y} \cdot \mathbf{k}'.$$

Let \mathbf{k}_{stat} be the stationary point of the phase-function:

$$\nabla g(\mathbf{k}_{stat}) = 0.$$

Then there exist $C_1 \in \mathbb{C}, C_2, C_3 \in \mathbb{R}$ so, that for all χ with $\|\partial_k^j \chi\|_s \leq \|\partial_k^j \tilde{\chi}\|_s$ for $j = 0, 1, 2$ and $\mathbf{y} \in \mathbb{R}^3$:

(a) For $a = 0$

$$\left\| \int e^{-i\mu g(\mathbf{k}')} \chi(\mathbf{k}') d^3 k' - C_1 \mu^{-\frac{3}{2}} \chi(\mathbf{k}_{stat}) \right\|_s < C_2 \mu^{-2}. \quad (16)$$

For phase functions without stationary point $C_1 = 0$, otherwise we can choose

$$C_1 = (-2\pi i)^{\frac{3}{2}} e^{-i\mu g(\mathbf{k}_{stat})} \frac{(k_{stat}^2 + m^2)^{\frac{5}{4}}}{m}.$$

(b) For $a > 0$

$$\left\| \int e^{-i\mu g(\mathbf{k}')} \chi(\mathbf{k}') d^3 k' \right\|_s < C_3 \left(\frac{k_{stat}}{\mu} \right)^{\frac{1}{2}} \|\chi(\mathbf{k}_{stat})\|_s + |C_1| \mu^{-\frac{3}{2}} \|\chi(\mathbf{k}_{stat})\|_s + C_2 \mu^{-2}. \quad (17)$$

For phase functions without stationary point $C_1 = C_3 = 0$

Moreover the C_j are uniformly bounded for all χ , a and \mathbf{y} .

In our application in the case $a \neq 0$, k_{stat} will be of order μ^{-1} so that the $|C_1|$ -term and the C_3 -term are of the same order.

This statement is a slight adaptation to our situation of a theorem of Hörmander [10], and its proof in the appendix 4.1.

3.3 Scattering into cones for a free particle

Applying Lemma 3.1 to (14) we choose:

$$\mu = \lambda\sqrt{k^2 + m^2}; \quad a = 0; \quad \mathbf{y} = \frac{\mathbf{k}}{\sqrt{k^2 + m^2}}; \quad \chi(\mathbf{k}') = (2\pi)^{-\frac{3}{2}}\widehat{\psi}(\mathbf{k}')$$

and calculate the stationary point k_{stat} :

$$\begin{aligned} \frac{\mathbf{k}_{stat}}{\sqrt{k_{stat}^2 + m^2}} - \mathbf{y} &= 0 \\ k_{stat}^2 &= y^2(k_{stat}^2 + m^2) \\ k_{stat} &= \frac{ym}{\sqrt{1 - y^2}} \end{aligned}$$

obtaining

Corollary 3.2 (*"Scattering into cones"*) *There exists a constant $C < \infty$ so that for all $\mathbf{k} \in \mathbb{R}^3$*

$$\left\| \psi(\lambda\mathbf{k}) - \frac{e^{-i\lambda m^2}}{(i\lambda)^{\frac{3}{2}}}\widehat{\psi}(\mathbf{k})\sqrt{\frac{k^2}{m^2} + 1} \right\|_s \leq C\lambda^{-2}.$$

Note, that this implies

$$\lim_{\lambda \rightarrow \infty} \sup_{\mathbf{k}} (\|\sqrt{\lambda^3}\psi(\lambda\mathbf{k})\|_s - \|\widehat{\psi}(\mathbf{k})\sqrt{\frac{k^2}{m^2} + 1}\|_s) = 0. \quad (18)$$

For the flux-across-surfaces theorem we need the asymptotics of the relativistic quantum flux (5) of the particle. Since all the α_l are bounded matrices and $\widehat{\psi} \in \mathcal{G}$, we obtain from (5) and (18) for the flux:

$$\lim_{\lambda \rightarrow \infty} \sup_{\mathbf{k}} | \lambda^3 j_l(\lambda \mathbf{k}) - \langle \widehat{\psi}(\mathbf{k}), \alpha_l \widehat{\psi}(\mathbf{k}) \rangle \left(\frac{k^2}{m^2} + 1 \right) | = 0 . \quad (19)$$

Next observe (see the appendix 4.2), that:

$$\langle \widehat{\psi}(\mathbf{k}), \alpha \widehat{\psi}(\mathbf{k}) \rangle = \frac{\mathbf{k}}{\sqrt{k^2 + m^2}} \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle . \quad (20)$$

Thus we get the uniform bound:

Corollary 3.3

$$\forall \varepsilon > 0 \quad \exists \lambda \in \mathbb{R} : \\ \sup_{\mathbf{k}} | \lambda^3 \mathbf{j}(\lambda \mathbf{k}) - \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle \frac{\mathbf{k}}{m^2} \sqrt{k^2 + m^2} | < \varepsilon . \quad (21)$$

Observe, that after a long time of propagation, the flux at $\mathbf{x} = \lambda \mathbf{k}$ will always be parallel to \mathbf{k} . So in the limit $t \rightarrow \infty$ it will always point away from the origin of the coordinate system.

3.4 Flux across surfaces for a free particle

Theorem 2.1 reads in this case

$$\lim_{R \rightarrow \infty} \int_S \int_0^\infty \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega - \int_S \int_0^\infty \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle k^2 dk d\Omega = 0 \quad (22)$$

and

$$\lim_{R \rightarrow \infty} \int_S \int_0^\infty j(\mathbf{R}, t) dt R^2 d\Omega - \int_S \int_0^\infty \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle k^2 dk d\Omega = 0 . \quad (23)$$

In the following, we will prove (22) by inserting the longtime asymptotic (21) for \mathbf{j} and showing, that the integral of the error we get by this approximation tends to zero in the limit $R \rightarrow \infty$.

Now, the long time asymptotic of \mathbf{j} is parallel to the normal \mathbf{n} of the radial surface. Therefore

the longtime asymptotic of j is equal to the longtime asymptotic of $\mathbf{j} \cdot \mathbf{n}$. More detailed, one sees that using the approximation (21) for \mathbf{j} in (22) and (23), the bound on the error terms in (22) and (23) arising from (21) are equal.

So the proof of (23) is essentially the same as for (22) and we shall concentrate only on showing (22).

The left side of (22) includes an integral over t , whereas the right hand side is integrated over k . We therefore substitute for t in the first term, to get integration over k , too. Since λ plays the role of a time parameter it is natural to substitute:

$$\mathbf{k} = \frac{R\mathbf{n}}{\lambda}$$

with

$$\lambda = \frac{\sqrt{t^2 - R^2}}{m} .$$

But this substitution is only possible in the time-like region ($t \geq R$). So we first handle the integral starting at $t = R$, later we deal with the space-like part of the integral. Then, substituting t by k , we obtain

$$\begin{aligned} \int_S \int_R^\infty \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega &= \int_S \int_0^\infty \mathbf{j}\left(\mathbf{R}, \frac{R}{k} \sqrt{k^2 + m^2}\right) \cdot \mathbf{n} \frac{m^2}{\sqrt{k^2 + m^2}} \frac{R^3}{k^2} dk d\Omega \\ &= \int_S \int_0^\infty \mathbf{j}(\lambda(k)k, \lambda(k)\sqrt{k^2 + m^2}) \cdot \mathbf{n} \frac{m^2}{\sqrt{k^2 + m^2}} k \lambda(k)^3 dk d\Omega . \end{aligned}$$

The integrand is now in the form that we can replace it by the asymptotic in (21).

It turns out however, that the error in the integrand will be $\sim \frac{k}{\sqrt{k^2 + m^2}}$ which is not integrable, therefore the replacement is not straight forward. We separate large momenta $k > X$ and small momenta $k < X$. In the following we choose $X > m$. Given X and $R_0 = \lambda_0 X$

$$k \leq X \Leftrightarrow \frac{R_0}{k} = \lambda(k) \geq \lambda_0 = \frac{R_0}{X} .$$

Then by (21) for small momenta ($k \leq X \Leftrightarrow t \geq R\sqrt{1 + \frac{m^2}{X^2}}$):

$$\forall \varepsilon > 0 \quad \exists R_0 \in \mathbb{R} \quad \forall R \geq R_0$$

$$\begin{aligned} & \left| \int_S \int_{R\sqrt{1+\frac{m^2}{X^2}}}^{\infty} (\mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 - \int_0^X \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle k^2) dk d\Omega \right| \\ &= \left| \int_S \int_0^X \mathbf{j}(\lambda k, \lambda \sqrt{k^2 + m^2}) \cdot \mathbf{n} \frac{m^2}{\sqrt{k^2 + m^2}} k \lambda^3 - \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle k^2 dk d\Omega \right| \\ &\leq \int_S \int_0^X \frac{km^2 \varepsilon}{\sqrt{k^2 + m^2}} dk d\Omega =: \chi(X) \varepsilon \end{aligned} \quad (24)$$

where

$$\chi(X) := 4\pi \int_0^X \frac{km^2}{\sqrt{k^2 + m^2}} dk.$$

Given X we can take ε arbitrarily small, choosing R_0 large enough, so that the r.h.s. of (24) goes to zero. Thus

$$\lim_{X \rightarrow \infty} \lim_{R \rightarrow \infty} \left| \int_S \int_{R\sqrt{1+\frac{m^2}{X^2}}}^{\infty} \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega - \int_S \int_0^X \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle k^2 dk d\Omega \right| = 0. \quad (25)$$

For the large momenta note that by virtue of $\widehat{\psi} \in \mathcal{G}$:

$$\lim_{X \rightarrow \infty} \int_S \int_X^{\infty} \langle \widehat{\psi}(\mathbf{k}), \widehat{\psi}(\mathbf{k}) \rangle k^2 dk d\Omega = 0 \quad (26)$$

and all it remains to show is that

$$\lim_{X \rightarrow \infty} \lim_{R \rightarrow \infty} \int_S \int_0^{R\sqrt{1+\frac{m^2}{X^2}}} \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega = 0 \quad (27)$$

where we also included the time integration outside the light cone, which we excluded in the substitution.

We first estimate the part of the integral (27) that lies in the space-like region (more precisely: $t \in [0, R]$) then we estimate the time-like part near the light cone ($t \in [R, R\sqrt{1+\frac{m^2}{X^2}}]$). That is, we first show that

$$\lim_{R \rightarrow \infty} \int \int_0^R \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega = 0. \quad (28)$$

That this holds is physically related to the fact, that a particle moves slower than light, so for big time and space scales the main part of the wavefunction will be inside the light cone. This follows from a straightforward application of the stationary phase method, outside of the stationary point, choosing a special coordinate system, where the k_1 -coordinate is along the direction \mathbf{x} . Two partial integrations lead to:

$$\begin{aligned} \|\psi(\mathbf{x}, \eta x)\|_s &= \left\| \int (2\pi)^{-\frac{3}{2}} e^{-ix(\sqrt{k^2+m^2}\eta-k_1)} \widehat{\psi}(\mathbf{k}) d^3k \right\|_s \\ &= \left\| \int (2\pi)^{-\frac{3}{2}} e^{-ixg} \widehat{\psi}(\mathbf{k}) d^3k \right\|_s \\ &\leq \frac{1}{x^2} \int \left\| (2\pi)^{-\frac{3}{2}} \left(\frac{\widehat{\psi}''}{g'^2} - \frac{3\widehat{\psi}'g''}{g'^3} + \frac{3\widehat{\psi}g''^2}{g'^4} - \frac{\widehat{\psi}g'''}{g'^3} \right) \right\|_s d^3k \end{aligned}$$

where

$$g := (\sqrt{k^2 + m^2}\eta - k_1) \quad f' := \partial_{k_1} f.$$

Since

$$-g' = 1 - \frac{k_1\eta}{\sqrt{k^2 + m^2}} \geq 1 - \frac{|k_1|}{\sqrt{k^2 + m^2}} > 0$$

it follows:

$$\|\psi(\mathbf{x}, \eta x)\|_s \leq (2\pi)^{-\frac{3}{2}} \frac{C_2}{x^2} \quad (29)$$

uniform in $\eta \leq 1$. Hence

$$\begin{aligned} \lim_{R \rightarrow \infty} \int \int_0^R \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega \\ \leq 4\pi \lim_{R \rightarrow \infty} \int_0^R \|\psi(\mathbf{x}, t)\|_s^2 dt R^2 \leq \frac{1}{2\pi^2} C_2^2 \lim_{R \rightarrow \infty} R^3 \frac{1}{R^4} = 0 \end{aligned}$$

It is left to prove that the second part of the integral in (27) goes to zero, i.e. that

$$\lim_{X \rightarrow \infty} \lim_{R \rightarrow \infty} \int_S \int_R^{R\sqrt{1+\frac{m^2}{X^2}}} \mathbf{j}(R, t) \cdot \mathbf{n} dt R^2 d\Omega = 0 .$$

The scalar norm of $\psi(\mathbf{x}, t)$ is:

$$\| \psi(\mathbf{x}, t) \|_s = \left\| \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t+i\mathbf{k}\cdot\mathbf{x}} \widehat{\psi} d^3k \right\|_s \quad (30)$$

$$= \left\| \int (2\pi)^{-\frac{3}{2}} e^{-i(\sqrt{k^2+m^2}-\mathbf{k}\cdot\mathbf{r})t} \widehat{\psi} d^3k \right\|_s . \quad (31)$$

Applying Lemma 3.1 with

$$\mu = t; \quad a = 0; \quad \mathbf{y} = \mathbf{r}; \quad \chi(\mathbf{k}') = (2\pi)^{-\frac{3}{2}} \widehat{\psi}(\mathbf{k}')$$

we have by (16), that:

$$\left\| \int e^{-iE_k t+i\mathbf{k}\cdot\mathbf{x}} \widehat{\psi}(\mathbf{k}) d^3k - C_1 t^{-\frac{3}{2}} \widehat{\psi}(k_{stat}) \right\|_s < C_2 t^{-2} .$$

As $\widehat{\psi}$ is bounded, we have:

$$\exists M \in \mathbb{R} : \forall t > R \quad \left\| \psi(\mathbf{x}, t) \right\|_s = \left\| \int e^{-iE_k t+i\mathbf{k}\cdot\mathbf{x}} \widehat{\psi}(\mathbf{k}) d^3k \right\|_s \leq M t^{-\frac{3}{2}} .$$

So

$$\left| \int_S \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} R^2 d\Omega \right| \leq 4\pi \frac{MR^2}{t^3} .$$

So we can write:

$$\begin{aligned}
& \left| \int_S \int_R^{\sqrt{1+\frac{m^2}{X^2}}} \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega \right| \\
& \leq 2\pi M R^2 (R^{-2} - R^{-2} (1 + \frac{m^2}{X^2})^{-1}) = 2\pi M (1 - (1 + \frac{m^2}{X^2})^{-1}).
\end{aligned}$$

This term goes to zero as $X \rightarrow \infty$

3.5 The flux-across-surfaces theorem with potential

3.5.1 Generalized Eigenfunctions for the Dirac equation with potential

For the proof of the free flux-across-surfaces theorem we used the $\varphi_{\mathbf{k}}^s$ as basis of the Hilbert space. In the potential case we adopt a new basis for doing calculations.

Like in the free case, we again get four linear independent eigenfunctions for each \mathbf{k} , two of them have positive energy-eigenvalue $E_k^{eig} = E_k = \sqrt{k^2 + m^2}$, two of them have negative energy-eigenvalue $E_k^{eig} = -E_k$. We denote by $\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ the eigenfunctions with $s \in \{1, 2\}$:

$$E_k \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) = (H_0 + A) \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}). \quad (32)$$

The corresponding Lipmann Schwinger equation reads:

$$\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) = \varphi_{\mathbf{k}}^s(\mathbf{x}) + (E_k - H_0)^{-1} A \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}). \quad (33)$$

We replace the formal expression $(E_k - H_0)^{-1}$ by the integral kernel G_k^+ :

$$(E_k - H_0) G_k^+(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \quad (34)$$

The explicit form for $G_k^+(\mathbf{x} - \mathbf{x}')$ can be found in [14]:

$$G_k^+(\mathbf{x}) = \frac{1}{4\pi} e^{ikx} \left(-x^{-1} (E_k + \sum_{j=1}^3 \alpha_j k \frac{x_j}{x} + \beta m) + x^{-2} \sum_{j=1}^3 \alpha_j \frac{x_j}{x} \right) =: \frac{e^{ikx}}{x} S_k^+(\mathbf{x}). \quad (35)$$

Thus:

$$\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) = \varphi_{\mathbf{k}}^s(\mathbf{x}) - \int A(\mathbf{x}') G_k^+(\mathbf{x} - \mathbf{x}') \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}') d^3 x' . \quad (36)$$

For $S_{\mathbf{k}}^+$, defined in (35), we have:

$$\begin{aligned} |\partial_k^j S_{\mathbf{k}}^+| &= \left| \frac{1}{4\pi} \partial_k^j \left(-E_k - \sum_{j=1}^3 \alpha_j k \frac{x_j}{x} - \beta m + x^{-1} \sum_{j=1}^3 \alpha_j \frac{x_j}{x} \right) \right| \\ &= \left| \frac{1}{4\pi} \partial_k^j \left(E_k + \sum_{j=1}^3 \alpha_j \left(k \frac{x_j}{x} - \frac{x_j}{x^2} \right) + \beta m \right) \right| \end{aligned}$$

for $j = 0, 1, 2$. For $x \geq 1$ we have

$$\left| \frac{x_j}{x} \right| \leq 1 \quad \left| \frac{x_j}{x^2} \right| \leq 1$$

and for such x , observing, that $\partial_k^j E_k \geq 0$ and k enters linearly in the second term, it follows, that

$$|\partial_k^j S_{\mathbf{k}}^+| \leq \left| \frac{1}{4\pi} \partial_k^j \left(E_k + \sum_{j=1}^3 \alpha_j (k + 1) + \beta m \right) \right|$$

Thus with

$$\tilde{S}_{\mathbf{k}}^+ := \frac{1}{4\pi} \left(E_k + \sum_{j=1}^3 \alpha_j (k + 1) + \beta m \right) \quad (37)$$

we have:

$$|\partial_k^j S_{\mathbf{k}}^+| \leq |\partial_k^j \tilde{S}_{\mathbf{k}}^+| \quad (38)$$

for $j = 0, 1, 2$, $x \geq 1$.

For the next steps we need some properties of the generalized eigenfunctions. We summarize these properties in the following Lemma which is proven in the Appendix 4.3:

Lemma 3.4 *Let A satisfy Condition A (3). Then there exist unique solutions $\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ of (36) for all $k \in \mathbb{R}^3$, such that:*

(a) For any $\mathbf{k} \in \mathbb{R}^3$, $s = 1, 2$ the functions $\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ are Hölder continuous of degree 1 in \mathbf{x}

(b) Any $\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ which is a solution of (36) automatically satisfies (32).

(c) The functions

$$\zeta_{\mathbf{k}}^s(\mathbf{x}) := \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) - \varphi_{\mathbf{k}}^s(\mathbf{x}) \quad (39)$$

are infinitely often continuously differentiable with respect to k , furthermore we have for $j \in \mathbb{N}$ and any multi-index γ with $|\gamma| \leq 2$:

$$\begin{aligned} i) \quad & \sup_{\mathbf{x} \in \mathbb{R}^3} \|x \zeta_{\mathbf{k}}^s(\mathbf{x})\|_s < \infty \\ ii) \quad & \sup_{\mathbf{x} \in \mathbb{R}^3} \left\| \partial_k^j \frac{\zeta_{\mathbf{k}}^s(\mathbf{x})}{|x+1|^{j-1}} \right\|_s < \infty \\ iii) \quad & \sup_{\mathbf{x} \in \mathbb{R}^3} \left\| k^{|\gamma|-1} D_{\mathbf{k}}^\gamma \frac{\zeta_{\mathbf{k}}^s(\mathbf{x})}{|x+1|^{j-1}} \right\|_s < \infty. \end{aligned}$$

(d) The $\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ form a basis of the space of scattering states, i.e. for scattering states $\psi(\mathbf{x}, t)$:

$$\psi(\mathbf{x}, t) = \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) \hat{\psi}_{out,s}(\mathbf{k}) d^3k \quad (40)$$

$$\hat{\psi}_{out,s}(\mathbf{k}) = \int (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}) \rangle d^3x \quad (41)$$

where $\hat{\psi}_{out,s}(\mathbf{k})$ is the fourier transform of $\psi_{out} = \Omega_+ \psi$.

3.5.2 Flux-across-surfaces for the Dirac-equation with potential

We prove now Theorem 2.1. As in the free case only the equality of the second and third integral is shown. From the nature of the estimates in the proof it will become evident, that essentially by the same argument as in the free case, the first equality can be established, and we do not say anything more to that.

We again split our flux integral into two parts, one inside the light-cone (from R to ∞) and one outside the light-cone (from 0 to R), where the main contribution comes from the times $t > R$, i.e. we prove that

$$\begin{aligned}
i) \quad & \lim_{R \rightarrow \infty} \left| \int_S \int_R^\infty \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega - \int_S \int_0^\infty \langle \widehat{\psi}_{\text{out}}(\mathbf{k}), \widehat{\psi}_{\text{out}}(\mathbf{k}) \rangle k^2 dk d\Omega \right| = 0 \\
ii) \quad & \lim_{R \rightarrow \infty} \int \int_0^R \mathbf{j}(\mathbf{R}, t) \cdot \mathbf{n} dt R^2 d\Omega = 0
\end{aligned} \tag{42}$$

We start with i):

According to (40)

$$\psi(\mathbf{x}, t) = \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) \widehat{\psi}_{\text{out},s}(\mathbf{k}) d^3k .$$

Setting

$$\widehat{\psi}_{\text{out}}(\mathbf{k}') = \sum_{s=1}^2 s_{\mathbf{k}'}^s \widehat{\psi}_{\text{out},s}(\mathbf{k}')$$

and using (36) with (39) we get:

$$\begin{aligned}
\psi(\mathbf{x}, t) &= \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} e^{i\mathbf{k}\cdot\mathbf{x}} \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3k \\
&\quad - \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} S_k^+(\mathbf{x}-\mathbf{x}') \mathcal{A}(\mathbf{x}') e^{i\mathbf{k}\cdot\mathbf{x}'} d^3x' \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3k \\
&\quad - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} S_k^+(\mathbf{x}-\mathbf{x}') \mathcal{A}(\mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3x' \widehat{\psi}_{\text{out},s}(\mathbf{k}) d^3k \\
&=: S_0 + S_1 + S_2 .
\end{aligned} \tag{43}$$

S_0 is the propagation of the free outgoing state. The free Flux-Across-Surfaces-Theorem yields therefore:

$$\lim_{R \rightarrow \infty} \left| \int_S \int_R^\infty \langle S_0, \alpha S_0 \rangle \cdot \mathbf{n} dt R^2 d\Omega - \int_S \int_0^\infty \langle \widehat{\psi}_{\text{out}}(\mathbf{k}), \widehat{\psi}_{\text{out}}(\mathbf{k}) \rangle k^2 dk d\Omega \right| = 0 .$$

Hence for (42)(i) it remains to show, that (using 5):

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \int_S \int_R^\infty (\mathbf{j}(R, t) - \langle S_0, \boldsymbol{\alpha} S_0 \rangle) \cdot \mathbf{n} dt R^2 d\Omega \\
&= \lim_{R \rightarrow \infty} \int_S \int_R^\infty (\langle \sum_{j=0}^2 S_j, \boldsymbol{\alpha} \sum_{j=0}^2 S_j \rangle - \langle S_0, \boldsymbol{\alpha} S_0 \rangle) \cdot \mathbf{n} dt R^2 d\Omega \\
&= \lim_{R \rightarrow \infty} \int_S \int_R^\infty (\langle \psi, \boldsymbol{\alpha} \sum_{j=1}^2 S_j \rangle + \langle \sum_{j=1}^2 S_j, \boldsymbol{\alpha} \psi \rangle) \cdot \mathbf{n} dt R^2 d\Omega = 0 .
\end{aligned}$$

By Schwartz-inequality we need only show:

$$\lim_{R \rightarrow \infty} \int_S \int_R^\infty \|\psi\|_s \sum_{j=1}^2 \|S_j\|_s dt R^2 d\Omega = 0 . \quad (44)$$

We first want to estimate $\|S_1\|_s$. Recalling (43) we get by Fubini's theorem:

$$\begin{aligned}
S_1 &= - \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} S_k^+(\mathbf{x}-\mathbf{x}') A(\mathbf{x}') e^{i\mathbf{k}\cdot\mathbf{x}'} d^3x' \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3k \\
&= - \int \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} S_k^+(\mathbf{x}-\mathbf{x}') A(\mathbf{x}') e^{i\mathbf{k}\cdot\mathbf{x}'} d^3x' \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3k \\
&=: - \int (2\pi)^{-\frac{3}{2}} \frac{1}{|\mathbf{x}-\mathbf{x}'|} \widetilde{S}_1(\mathbf{x}, \mathbf{x}') A(\mathbf{x}') d^3x'
\end{aligned}$$

where

$$\widetilde{S}_1(\mathbf{x}, \mathbf{x}') = \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} e^{ik|\mathbf{x}-\mathbf{x}'|} S_k^+(\mathbf{x}-\mathbf{x}') e^{i\mathbf{k}\cdot\mathbf{x}'} d^3x' \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3k . \quad (45)$$

Next we use Lemma 3.1, setting:

$$\mu = t; \quad a = t^{-1} |\mathbf{x} - \mathbf{x}'|; \quad \mathbf{y} = t^{-1} \mathbf{x}'; \quad k' = k; \quad \chi(\mathbf{k}') = (2\pi)^{-\frac{3}{2}} S_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \widehat{\psi}(\mathbf{k}') .$$

With regard to (38), the function

$$\widetilde{\chi}(\mathbf{k}) = (2\pi)^{-\frac{3}{2}} \widetilde{S}_{\mathbf{k}}^+ \widehat{\psi}(\mathbf{k}')$$

satisfies the properties we need in (17). Furthermore we observe that for the stationary point:

$$\begin{aligned}\frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} + a - y &= 0 \\ k_{stat} &= \sqrt{k_{stat}^2 + m^2}(y - a).\end{aligned}$$

So we can estimate k_{stat} by:

$$k_{stat} = \sqrt{k_{stat}^2 + m^2}t^{-1}(x' - |\mathbf{x} - \mathbf{x}'|) \leq \sqrt{k_{stat}^2 + m^2}xt^{-1}. \quad (46)$$

Hence by (17) we obtain for (45) that there exists $M_1 < \infty$, bounding in particular $\sqrt{k_{stat}^2 + m^2}\widehat{\chi}(\mathbf{k}_{stat})$, which is bounded by the choice of $\widehat{\psi}_{out} \in \mathcal{G}$ and incorporating also the constants C_1 and C_2 , uniformly in \mathbf{y} and a so that:

$$\begin{aligned}\|S_1\|_s &\leq \|M_1 t^{-\frac{3}{2}}(1 + x^{\frac{1}{2}}) \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} A(\mathbf{x}') d^3 x'\|_s \\ &= M_1 t^{-\frac{3}{2}} P_1(\mathbf{x}) \rightarrow_{x \rightarrow \infty} 0.\end{aligned} \quad (47)$$

That the function P_1 goes to zero in the limit $x \rightarrow \infty$ may be seen as follows:

For any function $f(\mathbf{x}) \in L^1$ with $\limsup_{x \rightarrow \infty} |x^3 f(\mathbf{x})| < \infty$ we have:

$$\begin{aligned}&\lim_{x \rightarrow \infty} x \left| \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} f(\mathbf{x}') d^3 x' \right| \\ &\leq \lim_{x \rightarrow \infty} x \int \left| \frac{1}{x'} f(\mathbf{x} - \mathbf{x}') \right| d^3 x' \\ &= \lim_{x \rightarrow \infty} x \left(\int_{B(0, \frac{x}{2})} \left| \frac{1}{x'} f(\mathbf{x} - \mathbf{x}') \right| d^3 x' + \int_{\mathbb{R}^3 \setminus B(0, \frac{x}{2})} \left| \frac{1}{x'} f(\mathbf{x} - \mathbf{x}') \right| d^3 x' \right) \\ &\leq \lim_{x \rightarrow \infty} x \left(\sup_{\tilde{x} \geq \frac{x}{2}} \{ |f(\tilde{\mathbf{x}})| \} \int_{B(0, \frac{x}{2})} \frac{1}{x'} d^3 x' + \frac{2}{x} \int_{\mathbb{R}^3 \setminus B(0, \frac{x}{2})} |f(\mathbf{x} - \mathbf{x}')| d^3 x' \right) \\ &\leq \lim_{x \rightarrow \infty} \frac{1}{8} x^3 \sup_{\tilde{x} \geq \frac{x}{2}} \{ |f(\tilde{\mathbf{x}})| \} + \lim_{x \rightarrow \infty} 2 \int_{\mathbb{R}^3 \setminus B(0, \frac{x}{2})} |f(\mathbf{x} - \mathbf{x}')| d^3 x' < \infty\end{aligned} \quad (48)$$

where $B(\mathbf{a}, r)$ means the ball with center \mathbf{a} and radius r .

Next we estimate $\|S_2\|_s$. According to (43) we can write it down as:

$$S_2 = - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}t} \int \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} S_k^+(\mathbf{x}-\mathbf{x}')(x'+1)A(\mathbf{x}') \frac{\zeta_{\mathbf{k}}^s(\mathbf{x}')}{x'+1} d^3x' \widehat{\psi}_{out,s}(\mathbf{k}) d^3k .$$

Therefore we again use Lemma 3.1, setting:

$$\mu = t; \quad a = t^{-1}(|\mathbf{x}-\mathbf{x}'|); \quad \mathbf{y} = 0; \quad k' = k; \quad \chi(\mathbf{k}') = (2\pi)^{-\frac{3}{2}} \sum_{s=1}^2 \frac{\zeta_{\mathbf{k}}^s(\mathbf{x}')}{x'+1} S_k^+(\mathbf{x}-\mathbf{x}') \widehat{\psi}_{out,s}(\mathbf{k}') .$$

With regard to (38) and Lemma 3.4(c) there exists a $M_2 < \infty$, so that the function

$$\widetilde{\chi} = (2\pi)^{-\frac{3}{2}} M_2 \widetilde{S}_{\mathbf{k}}^+ \widehat{\psi}(\mathbf{k}')$$

satisfies the properties we need in Lemma 3.1.

Since our phase function has no stationary point we get with (17):

$$\|S_2\|_s \leq M_2 t^{-2} \left| \int \frac{1}{|\mathbf{x}-\mathbf{x}'|} (x'+1)A(\mathbf{x}') d^3x' \right| = M_2 t^{-2} P_2(\mathbf{x}) \xrightarrow{x \rightarrow \infty} 0 .$$

Choosing $(x'+1)A(\mathbf{x}')$ for f in the most left side of (48), one can see, that $xP_2(\mathbf{x})$ is bounded, so P_2 goes to zero in the limit $x \rightarrow \infty$. Since S_0 is the analogue of the freely evolving wavefunction, we have by Corollary 3.2:

$$\|S_0\|_s \leq M_0 t^{-\frac{3}{2}} . \tag{49}$$

We use the estimates (49), (47) and (49) in the right side of (44) and get, defining $M := M_0 + M_1 + M_2$:

$$\begin{aligned}
& \lim_{R \rightarrow \infty} \left| \int_S \int_R^\infty (\|\psi\|_s \|\sum_{j=1}^2 S_j\|_s) dt R^2 d\Omega \right| \\
& \leq \lim_{R \rightarrow \infty} \int_R^\infty M^2(P_1(\mathbf{R}) + P_2(\mathbf{R})) t^{-3} dt R^2 \leq \lim_{R \rightarrow \infty} 3M^2(P_1(\mathbf{R}) + P_2(\mathbf{R})) = 0
\end{aligned}$$

and (44) is proved.

Like in the free case, (42 ii) follows directly from an analogous argument which used equation (29), thus we prove (29) for the case at hand. Since in (42 ii) we need only estimates of the wavefunction for times $t \leq x$ we have in view of (43), setting $t = \eta x$ with $0 \leq \eta \leq 1$ and using Fubini's Theorem:

$$\begin{aligned}
\psi(\mathbf{x}, \eta x) &= \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}\eta x + i\mathbf{k}\cdot\mathbf{x}} \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3 k \\
&\quad - \int \int e^{-i\sqrt{k^2+m^2}\eta x + ik|\mathbf{x}-\mathbf{x}'| + i\mathbf{k}\cdot\mathbf{x}'} \frac{A(\mathbf{x}') S_{\mathbf{k}}^+(\mathbf{x}-\mathbf{x}') \widehat{\psi}_{\text{out}}(\mathbf{k})}{(2\pi)^{\frac{3}{2}} |\mathbf{x}-\mathbf{x}'|} d^3 k d^3 x' \\
&\quad - \sum_{s=1}^2 \int \int e^{-i\sqrt{k^2+m^2}\eta x + ik|\mathbf{x}-\mathbf{x}'|} \frac{A(\mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') S_{\mathbf{k}}^+(\mathbf{x}-\mathbf{x}') \widehat{\psi}_{\text{out},s}(\mathbf{k})}{(2\pi)^{\frac{3}{2}} |\mathbf{x}-\mathbf{x}'|} d^3 k d^3 x' \\
&=: S_0 + S_1 + S_2 .
\end{aligned}$$

For S_0 we have (29), for the other summands we define:

$$\begin{aligned}
\widetilde{S}_1 &:= \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}\eta x + ik|\mathbf{x}-\mathbf{x}'| + i\mathbf{k}\cdot\mathbf{x}'} S_{\mathbf{k}}^+(\mathbf{x}-\mathbf{x}') \widehat{\psi}_{\text{out}}(\mathbf{k}) d^3 k \\
\widetilde{S}_2 &:= \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} e^{-i\sqrt{k^2+m^2}\eta x + ik|\mathbf{x}-\mathbf{x}'| + i\mathbf{k}\cdot\mathbf{x}'} e^{-i\mathbf{k}\cdot\mathbf{x}'} \zeta_{\mathbf{k}}^s(\mathbf{x}') S_{\mathbf{k}}^+(\mathbf{x}-\mathbf{x}') \widehat{\psi}_{\text{out},s}(\mathbf{k}) d^3 k .
\end{aligned}$$

So we have for S_j , $j=1;2$:

$$S_j = - \int \widetilde{S}_j \frac{A(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3 x' .$$

We can estimate the \widetilde{S}_j by two partial integrations. One can easily see, that the phase functions of \widetilde{S}_j have no stationary point. This leads to:

$$\begin{aligned}
\| \tilde{S}_j \|_s &= \left\| \int (2\pi)^{-\frac{3}{2}} e^{-ixg(\mathbf{k})} \chi_j(\mathbf{x}, \mathbf{x}', \mathbf{k}) d^3k \right\|_s \\
&= \frac{1}{x^2} \left\| \int (2\pi)^{-\frac{3}{2}} e^{-ixg(\mathbf{k})} \partial_{k_1} \left(\frac{1}{g'} \partial_{k_1} \frac{\chi_j}{g'} \right) d^3k \right\|_s \\
&= \frac{1}{x^2} \left\| \int (2\pi)^{-\frac{3}{2}} \left(\frac{\chi_j''}{g'^2} - \frac{3\chi_j' g''}{g'^3} + \frac{3\chi_j g''^2}{g'^4} \right) d^3k \right\|_s
\end{aligned}$$

where

$$\begin{aligned}
g(\mathbf{k}) &:= \sqrt{k^2 + m^2} \eta - k \frac{|\mathbf{x} - \mathbf{x}'|}{x} - \mathbf{k} \cdot \frac{\mathbf{x}'}{x} \\
\chi_1(\mathbf{x}, \mathbf{x}', \mathbf{k}) &:= S_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \widehat{\psi}_{\text{out}}(\mathbf{k}) \\
\chi_2(\mathbf{x}, \mathbf{x}', \mathbf{k}) &:= \sum_{s=1}^2 e^{-i\mathbf{k} \cdot \mathbf{x}'} \zeta_{\mathbf{k}}^s(\mathbf{x}') S_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \widehat{\psi}_{\text{out},s}(\mathbf{k}) \\
g' &:= \partial_{k_1} g.
\end{aligned}$$

Since

$$\begin{aligned}
|g'| &= \frac{x'}{x} + \frac{k_1 |\mathbf{x} - \mathbf{x}'|}{kx} - \frac{k_1 \eta}{\sqrt{k^2 + m^2}} \geq \frac{k_1}{k} \left(\frac{x'}{x} + \frac{|\mathbf{x} - \mathbf{x}'|}{x} - \frac{k}{\sqrt{k^2 + m^2}} \right) \\
&\geq \frac{k_1}{k} \left(1 - \frac{k}{\sqrt{k^2 + m^2}} \right) > 0.
\end{aligned}$$

g'' is bounded and due to Lemma 3.4 the χ_j are bounded, we can find $C_2 < \infty$ with:

$$\sum_{j=1}^2 \tilde{S}_j \leq \frac{C_2}{x^2}.$$

So $x^2 \sum_{j=1}^2 S_j$ is bounded (see 48) and the analogue of (29) is proved.

4 Appendix

4.1 Proof of Lemma(3.1)

We consider for a family of phase functions g , which we should think of being indexed by $a \geq 0, \mathbf{y}$:

$$g(\mathbf{k}) = \sqrt{k^2 + m^2} + a |k| - \mathbf{y} \cdot \mathbf{k}$$

the integral

$$I := \int e^{-i\mu g(\mathbf{k})} \chi(\mathbf{k}) d^3k$$

where $\chi \in \mathcal{G}$ (see7).

We shall find its asymptotic behavior as a function of μ . In major parts we will recall the proof of theorem 7.7.5 in the book of Hörmander [10], which unfortunately is formulated for compactly supported χ and which moreover does not give uniformity over the family, i.e. uniformity in a, \mathbf{y} which we need. The compactness can easily be handled but for the uniformity we must invoke the special form of the family of phase functions g and we shall give the argument here.

The stationary points of the phase functions are given by:

$$\begin{aligned} g'(\mathbf{k}_{stat}) &= \frac{\mathbf{k}_{stat}}{\sqrt{k_{stat}^2 + m^2}} + a \frac{\mathbf{k}_{stat}}{k_{stat}} - \mathbf{y} = 0 \\ k_{stat}^2 &= (k_{stat}^2 + m^2)(y - a)^2 \\ k_{stat} &= \frac{m(y - a)}{\sqrt{1 - (y - a)^2}} \\ \mathbf{k}_{stat} &\parallel \mathbf{y}. \end{aligned} \tag{50}$$

Since k_{stat} is a function of a and \mathbf{y} , we sometimes use the phrase: uniform in k_{stat} to express uniformity in a and \mathbf{y} . (I) For $y \geq a+1$ there is no stationary point and for $y = a$ the stationary point is at $k_{stat} = 0$.

First we handle the family where $y \in [a + \frac{1}{2}; a + 1[$. These phase-functions do exactly have one stationary point bounded away from zero:

$$k_{stat} = \frac{m(y-a)}{\sqrt{1-(y-a)^2}} \geq \frac{m}{\sqrt{3}} \quad (51)$$

Later we will handle phase functions, where the stationary point is close to zero and phase-functions without stationary point.

We choose a coordinate system, where the k_1 -direction is parallel to \mathbf{y} . So the stationary points will have the coordinates $(k_{stat}, 0, 0)$. To estimate the integral, we separate from the integral the contribution coming from near the stationary point. This part of integral includes the leading term. Therefore we define a smooth function $\rho_{k_{stat}}$ which is one near the stationary points and zero away from the stationary point. (We shall omit further on for ease of notation the index k_{stat}).

More precisely we define the compact set Q by:

$$\mathbf{k} \in Q \Leftrightarrow k_1 \in [\frac{k_{stat}}{2}, 2k_{stat}] \wedge k_2, k_3 \in [-1, 1]$$

and choose

$$\rho(\mathbf{k}) := 1 \quad \forall \mathbf{k} \in Q \quad (52)$$

falling quickly off to zero outside of Q , lets say

$$\rho(\mathbf{k}) := 0 \quad \forall \mathbf{k} \notin Q_\varepsilon \quad (53)$$

where Q_ε is some ε -neighborhood of Q for some $\varepsilon > 0$. With the help of ρ we can split $\chi = \chi_1 + \chi_2$ by defining:

$$\begin{aligned} \chi_1 &:= \rho\chi & \chi_2 &:= (1 - \rho)\chi \\ I_1 &:= \int e^{-i\mu g(\mathbf{k})} \chi_1(\mathbf{k}) d^3k & I_2 &:= \int e^{-i\mu g(\mathbf{k})} \chi_2(\mathbf{k}) d^3k . \end{aligned}$$

This split has the following advantages:

The compactly supported χ_1 includes the stationary point, so I_1 can be estimated the same

way as in Hörmanders theorem, but with focus on the uniformity of the estimates. χ_2 is zero near the stationary point, so I_2 can be easily estimated by partial integrations. ρ has been defined in such a way, that we may estimate the terms we get by the partial integrations uniform in k_{stat} .

We start with I_1 . We move the stationary point to the center of our coordinate system setting $\mathbf{k}' := \mathbf{k} - \mathbf{k}_{stat}$, i.e. $g(\mathbf{k})$ becomes $\tilde{g}(\mathbf{k}') = g(\mathbf{k}' + \mathbf{k}_{stat})$. Slightly abusing notation we simply write $g(\mathbf{k}')$ for \tilde{g} . By Taylor's formula we obtain a function f :

$$g(\mathbf{k}') = g(\mathbf{k}' = 0) + \sum_{|\gamma|=2} \frac{D_{\mathbf{k}'}^\gamma g(\mathbf{k}' = 0) \mathbf{k}'^\gamma}{\gamma!} + f(\mathbf{k}'), \quad (54)$$

where $\frac{f(\mathbf{k}')}{k'^3}$ bounded.

Computing the second-order terms of $g(\mathbf{k}')$ we find that only diagonal terms survive at $(k_{stat}, 0, 0)$ and

$$\begin{aligned} \partial_{k'_j}^2 g(\mathbf{k}' = 0) &= \partial_{k_j}^2 g(\mathbf{k} = \mathbf{k}_{stat}) \\ &= (\partial_{k_j} (\frac{k_j}{\sqrt{k^2 + m^2}} + a \frac{k_j}{k} - y_l)) |_{\mathbf{k}=\mathbf{k}_{stat}} \\ &= (\frac{k^2 - k_j^2 + m^2}{\sqrt{k^2 + m^2}^3} + a \frac{k^2 - k_j^2}{k^3}) |_{\mathbf{k}=\mathbf{k}_{stat}} \end{aligned} \quad (55)$$

so that

$$\begin{aligned} \partial_{k'_j}^2 g(\mathbf{k}' = 0) &= \frac{k_{stat}^2 + m^2}{\sqrt{k_{stat}^2 + m^2}^3} + a \frac{1}{k_{stat}} \text{ for } j = 2, 3 \\ \partial_{k'_1}^2 g(\mathbf{k}' = 0) &= \frac{m^2}{\sqrt{k_{stat}^2 + m^2}^3}. \end{aligned} \quad (56)$$

We define:

$$g_2(\vartheta, \theta) := \frac{\sum_{j=1}^3 \partial_{k'_j}^2 g(\mathbf{k}' = 0) k_j'^2}{k'^2}. \quad (57)$$

By this definition, g_2 does only depend on the angular, not on the radial coordinate of \mathbf{k}' . Using (57) in (54), we may write

$$g(\mathbf{k}') = g(0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta) + f(\mathbf{k}') . \quad (58)$$

Furthermore for $s \in [0, 1]$ set:

$$g_s := g(0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta) + s f(\mathbf{k}') \quad (59)$$

and

$$I(s) = \int e^{-i\mu g_s(\mathbf{k}')} \chi_1(\mathbf{k}') d^3 k' .$$

Note that $g = g_1$, $I_1 = I(1)$. By Taylor's Formula there exists $\xi \leq 1$ so that:

$$I_1 = I(1) = I(0) + \partial_s I(s) |_{\xi} . \quad (60)$$

We begin with $I(0)$, introducing spherical coordinates. With slight abuse of notation: (leaving the notation for the functions unchanged)

$$I(0) = \int e^{-i\mu(g(0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta))} \chi_1(k', \vartheta, \theta) k'^2 dk' d\Omega .$$

Writing $\chi_1 = \chi(k' = 0) + \tilde{\chi}$ the integral splits into:

$$\begin{aligned} I(0) &= \int e^{-i\mu(g(0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta))} \chi(k' = 0) k'^2 dk' d\Omega \\ &+ \int e^{-i\mu(g(0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta))} \tilde{\chi}(k', \vartheta, \theta) k'^2 dk' d\Omega =: I_1^1 + I_1^2 . \end{aligned} \quad (61)$$

The integral I_1^1 is a gaussian integral, which includes the leading term:

$$\begin{aligned}
I_1^1 &= \int e^{-i\mu(g(0)+\frac{1}{2}k'^2 g_2(\vartheta,\theta))} \chi(k'=0) k'^2 dk' d\Omega \\
&= \int e^{-i\mu \sum_{j=1}^3 \frac{1}{2} \partial_{k'_j}^2 g(\mathbf{k}'=0)} k_j^2 e^{-i\mu g(0)} \chi(\mathbf{k}'=0) k'^2 d^3 k' \\
&= (2\pi)^{\frac{3}{2}} \mu^{-\frac{3}{2}} e^{-i\mu g(0)} \left(\prod_{j=1}^3 \partial_{k'_j}^2 g(\mathbf{k}'=0) \right)^{-\frac{1}{2}} \chi(\mathbf{k}_{stat}) .
\end{aligned} \tag{62}$$

For $a = 0$ the $\partial_{k'_j}^2 g(\mathbf{k}' = 0)$ terms can be easily calculated. We get:

$$\begin{aligned}
\partial_{k'_j}^2 g(\mathbf{k}' = 0) &= \partial_{k'_j}^2 g(\mathbf{k} = \mathbf{k}_{stat}) \\
&= \partial_{k'_j} \frac{k_j}{\sqrt{k^2 + m^2}} \Big|_{\mathbf{k}=\mathbf{k}_{stat}} = \frac{k^2 + m^2 - k_j^2}{\sqrt{k^2 + m^2}^3} \Big|_{\mathbf{k}=\mathbf{k}_{stat}} .
\end{aligned}$$

So we get:

$$\prod_{j=1}^3 \partial_{k'_j}^2 g(\mathbf{k}' = 0) = \frac{m^2 (k_{stat}^2 + m^2)^2}{\sqrt{k_{stat}^2 + m^2}^9} = \frac{m^2}{\sqrt{k_{stat}^2 + m^2}^5} .$$

I_1^1 is the leading term of our integral. For $a = 0$ we get the desired value for C_1 (3.1).

For I_1^2 put:

$$\phi(k', \vartheta, \theta) := \tilde{\chi}(k', \vartheta, \theta) k'^{-1}$$

which is bounded and smooth.

$$I_1^2 = \int e^{-i\mu(g(0)+\frac{1}{2}k'^2 g_2(\vartheta,\theta))} \phi(k', \vartheta, \theta) k'^3 dk' d\Omega . \tag{63}$$

One partial integration leads to:

$$\begin{aligned}
\| I_1^2 \|_s &= \mu^{-1} \left\| \int e^{-i\mu \frac{1}{2} k'^2 g_2(\vartheta,\theta)} \partial_{k'} \frac{\phi(k', \vartheta, \theta) k'^3}{k' g_2(\vartheta, \theta)} dk' d\Omega \right\|_s \\
&= \mu^{-1} \left\| \int e^{-i\mu \frac{1}{2} k'^2 g_2(\vartheta,\theta)} \frac{\partial_{k'} \phi(k', \vartheta, \theta) k'^2 + 2\phi(k', \vartheta, \theta) k'}{g_2(\vartheta, \theta)} dk' d\Omega \right\|_s .
\end{aligned}$$

So another partial integration is possible:

$$\begin{aligned}
\| I_1^2 \|_s &= \mu^{-2} \left\| \int e^{-i\mu \frac{1}{2} k'^2 g_2(\vartheta, \theta)} \partial_{k'} \left(\frac{\partial_{k'} \phi(k', \vartheta, \theta) k'^2 + 2\phi(k', \vartheta, \theta) k'}{k' (g_2(\vartheta, \theta))^2} \right) dk' d\Omega \right\|_s \\
&= \mu^{-2} \left\| \int e^{-i\mu \frac{1}{2} k'^2 g_2(\vartheta, \theta)} \partial_{k'} \left(\frac{\partial_{k'} \phi(k', \vartheta, \theta) k' + 2\phi(k', \vartheta, \theta)}{(g_2(\vartheta, \theta))^2} \right) dk' d\Omega \right\|_s \\
&\leq \mu^{-2} \left\| \int \partial_{k'} \left(\frac{\partial_{k'} \phi(k', \vartheta, \theta) k' + 2\phi(k', \vartheta, \theta)}{(g_2(\vartheta, \theta))^2} \right) dk' d\Omega \right\|_s .
\end{aligned} \tag{64}$$

With our definition of Q , the support of the integrand increases and $g_2(\vartheta, \theta)$ decreases polynomially with k_{stat} (see (56) and (57)). While the support moves away from the center of our coordinate system. But $\tilde{\chi} = \chi - \chi(\mathbf{k}_{stat})$ and its derivatives decay faster in k_{stat} than any power, so we get a constant C uniform in k_{stat} with:

$$I_1^2 \leq \mu^{-2} C .$$

For I_1 it is left to estimate $\partial_s I(s) |_\xi$:

$$\partial_s I(s) |_\xi = \int -i\mu f(k', \vartheta, \theta) e^{-i\mu g_\xi(k', \vartheta, \theta)} \chi_1(k', \vartheta, \theta) k'^2 dk' d\Omega . \tag{65}$$

By Taylor's formula we can define:

$$\tilde{f}(k', \vartheta, \theta) := f(k', \vartheta, \theta) k'^{-3} \quad \tilde{g}(k', \vartheta, \theta) := k'^{-1} \partial_{k'} g_\xi(k', \vartheta, \theta)$$

and thus:

$$\partial_s I(s) |_\xi = \int -i\mu \tilde{f}(k', \vartheta, \theta) e^{-i\mu g_\xi(k', \vartheta, \theta)} \chi_1(k', \vartheta, \theta) k'^5 dk' d\Omega . \tag{66}$$

On Q_ε (see below (53)), g is infinitely often differentiable. So these functions are well defined and bounded on Q_ε .

To estimate the integral by partial integrations we have to assure, that g_ξ has only one stationary point, which is $k_{stat} = 0$ as one easily sees from (67).

By (59):

$$g_\xi = g(k' = 0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta) + \xi f(\mathbf{k}') = \xi g + (1 - \xi) \left(g(k' = 0) + \frac{1}{2}k'^2 g_2(\vartheta, \theta) \right). \quad (67)$$

Looking at

$$\partial_{k'}^2 g_\xi = \xi \partial_{k'}^2 g + (1 - \xi) \partial_{k'}^2 \frac{1}{2} k'^2 g_2$$

we observe, that

$$\begin{aligned} \partial_{k'}^2 g &= \partial_{k'}^2 \left(\sqrt{k'^2 - 2k'k_{stat} \cos(\vartheta) + k_{stat}^2 + m^2} + a \sqrt{k'^2 - 2k'k_{stat} \cos(\vartheta) + k_{stat}^2} - \mathbf{y} \cdot \mathbf{k}' \right) \\ &= \partial_{k'} \left(\frac{k' - k_{stat} \cos(\vartheta)}{\sqrt{k'^2 - 2k'k_{stat} \cos(\vartheta) + k_{stat}^2 + m^2}} + a \frac{k' - k_{stat} \cos(\vartheta)}{\sqrt{k'^2 - 2k'k_{stat} \cos(\vartheta) + k_{stat}^2}} \right) \\ &= \frac{(1 - \cos(\vartheta)^2)k_{stat}^2 + m^2}{\sqrt{k'^2 - 2k'k_{stat} \cos(\vartheta) + k_{stat}^2 + m^2}^3} + a \frac{(1 - \cos(\vartheta)^2)k_{stat}^2}{\sqrt{k'^2 - 2k'k_{stat} \cos(\vartheta) + k_{stat}^2}^3} > 0. \end{aligned}$$

And for $\mathbf{k} \in Q_\varepsilon$, k_1 is positive, so the angular component $\vartheta \in] - \frac{\pi}{2}, \frac{\pi}{2}[$, we also have, that on Q_ε also g_2 is positive. Since $\xi \in [0; 1]$ it follows, that $\partial_{k'}^2 g_\xi$ is positive, so $\partial_{k'} g_\xi$ is strictly monotonous on Q_ε and has only one stationary point. Recalling the definition of \tilde{g} (see 66) we see, that \tilde{g} is bounded away from zero.

Now we can estimate the integral (66). By partial integration

$$\begin{aligned} \partial_s I(s) |_\xi &= \int e^{-i\mu g_\xi(k', \vartheta, \theta)} \partial_{k'} \frac{\tilde{f}(k', \vartheta, \theta) \chi_1(k', \vartheta, \theta) k'^4}{\tilde{g}(k', \vartheta, \theta)} dk' d\Omega \\ &= \int e^{-i\mu g_\xi(k', \vartheta, \theta)} \left(\partial_{k'} \frac{\tilde{f}(k', \vartheta, \theta) \chi_1(k', \vartheta, \theta)}{\tilde{g}(k', \vartheta, \theta)} k'^4 + 4 \frac{\tilde{f}(k', \vartheta, \theta) \chi_1(k', \vartheta, \theta)}{\tilde{g}(k', \vartheta, \theta)} \right) k'^3 dk' d\Omega. \end{aligned}$$

Setting

$$\tilde{\psi}(k', \vartheta, \theta) := \partial_{k'} \frac{\tilde{f}(k', \vartheta, \theta) \chi_1(k', \vartheta, \theta)}{\tilde{g}(k', \vartheta, \theta)} k'^4 + 4 \frac{\tilde{f}(k', \vartheta, \theta) \chi_1(k', \vartheta, \theta)}{\tilde{g}(k', \vartheta, \theta)}. \quad (68)$$

Hence

$$\partial_s I(s) |_{\xi} = \int e^{-i\mu g_{\xi}(k', \vartheta, \theta)} \tilde{\psi}(k', \vartheta, \theta) k'^3 dk' d\Omega .$$

This term is similar to (63). The only differences are, that we have $\tilde{\psi}$ instead of ϕ and g_{ξ} instead of g_0 .

So with the same estimate as in (63) we get:

$$\| \partial_s I(s) |_{\xi} \|_s \leq \mu^{-2} \left\| \int \partial_{k'} \frac{\partial_{k'} \tilde{\psi}(k', \vartheta, \theta) k' + 2\tilde{\psi}(k', \vartheta, \theta)}{(\tilde{g}(\mathbf{k}', \vartheta, \theta))^2} dk' d\Omega \right\|_s . \quad (69)$$

This term again has uniform bound in k_{stat} , as its support moves away from the center of the coordinate system. So we get a constant C uniform in k_{stat} with:

$$\| \partial_s I(s) |_{\xi} \|_s \leq \mu^{-2} C .$$

Now we estimate $I(2)$ (54). As this integral includes no stationary point, two partial integrations are possible without any problem, but we have to assure, that we can estimate the factors we get by these partial integrations uniform in k_{stat} . To be able to find an uniform estimate, we estimate the areas of χ separately.

So we again split our integral:

$$\begin{aligned} I_2 &= \int_{k_1 < \frac{k_{stat}}{2}} e^{-i\mu g(\mathbf{k})} \chi_2(\mathbf{k}) d^3 k + \int_{k_1 > 2k_{stat}} e^{-i\mu g(\mathbf{k})} \chi_2(\mathbf{k}) d^3 k \\ &\quad + \int_{k_1 \in B; |k_2| > 1} e^{-i\mu g(\mathbf{k})} \chi_2(\mathbf{k}) d^3 k + \int_{k_1 \in B; |k_2| < 1; |k_3| > 1} e^{-i\mu g(\mathbf{k})} \chi_2(\mathbf{k}) d^3 k \\ &=: I_2^1 + I_2^2 + I_2^3 + I_2^4 \end{aligned}$$

where $B := [\frac{k_{stat}}{2}; 2k_{stat}]$.

The integrals I_2^1 and I_2^2 we estimate by two partial integrations under the k_1 -integral. This leads to:

$$\begin{aligned}
\| I_2^1 \|_s &\leq \mu^{-2} \int_{k_1 < \frac{k_{stat}}{2}} \left\| \partial_{k_1} \left(\frac{1}{\dot{g}(\mathbf{k})} \partial_{k_1} \frac{\chi_2(\mathbf{k})}{\dot{g}(\mathbf{k})} \right) \right\|_s d^3 k \\
&= \mu^{-2} \int_{k_1 < \frac{k_{stat}}{2}} \left\| 3 \frac{\ddot{\chi}_2}{\dot{g}^2} + 3 \frac{\chi_2 \ddot{g}^2}{\dot{g}^4} - 3 \frac{\dot{\chi}_2 \ddot{g}}{\dot{g}^3} \right\|_s d^3 k \\
\| I_2^2 \|_s &\leq \mu^{-2} \int_{k_1 > 2k_{stat}} \left\| \partial_k \left(\frac{1}{g'(\mathbf{k})} \partial_k \frac{\chi_2(\mathbf{k})}{g'(\mathbf{k})} \right) \right\|_s d^3 k \\
&= \mu^{-2} \int_{k_1 > 2k_{stat}} \left\| 3 \frac{\chi_2''}{g'^2} + 3 \frac{\chi_2 g''^2}{g'^4} - 3 \frac{\chi_2' g''}{g'^3} \right\|_s d^3 k
\end{aligned} \tag{70}$$

where $\dot{g}(\mathbf{k}) := \partial_{k_1} g(\mathbf{k})$; $g'(\mathbf{k}) := \partial_k g(\mathbf{k})$

At first sight these estimates do not seem to be uniform in a and \mathbf{y} . In fact

$$\ddot{g}(\mathbf{k}) = \frac{m^2}{\sqrt{k^2 + m^2}^3} + a \frac{k_2^2 + k_3^2}{k^3}$$

and

$$g''(\mathbf{k}) = \frac{m^2}{\sqrt{k^2 + m^2}^3}$$

are bounded on the area of integration. So it is left to show, that we can find functions h_j with $j=1;2$, which do not depend on a and \mathbf{y} and which is bounded away from zero on $\mathbb{R}^3 \setminus Q$ with

$$h_1(\mathbf{k}) \leq g'(\mathbf{k}) \quad h_2(\mathbf{k}) \leq \dot{g}(\mathbf{k})$$

for all $a, \mathbf{y}, \mathbf{k}$.

For this we estimate \dot{g} for $k_1 \leq \frac{k_{stat}}{2}$. As $\dot{g} > 0$, it follows, that (see (61))

$$|\dot{g}(\mathbf{k})| = y - \frac{k_1}{\sqrt{k^2 + m^2}} - a \frac{k_1}{k} \geq \frac{1}{2}.$$

For $k_1 < 0$ and by virtue $y \geq a + \frac{1}{2} \geq \frac{1}{2}$.

For $k_1 > 0$ we estimate, using that $y - a - \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} = 0$:

$$\begin{aligned}
|\dot{g}(\mathbf{k})| &= y - \frac{k_1}{\sqrt{k^2 + m^2}} - a \frac{k_1}{k} \geq y - a - \frac{k_1}{\sqrt{k^2 + m^2}} \\
&\geq y - a - \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} + \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} - \frac{k_1}{\sqrt{k^2 + m^2}} \\
&= \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} - \frac{k_1}{\sqrt{k^2 + m^2}} \\
&\geq \frac{k_{stat}\sqrt{k^2 + m^2} - k_1\sqrt{k_{stat}^2 + m^2}}{\sqrt{k^2 + m^2}\sqrt{k_{stat}^2 + m^2}} \\
&= \frac{k_{stat}^2(k^2 + m^2) - k_1^2(k_{stat}^2 + m^2)}{\left(k_{stat}\sqrt{k^2 + m^2} + k_1\sqrt{k_{stat}^2 + m^2}\right)\sqrt{k_1^2 + m^2}\sqrt{k_{stat}^2 + m^2}}.
\end{aligned}$$

Recalling $k \in [0, \frac{k_{stat}}{2}]$

$$\begin{aligned}
|\dot{g}(\mathbf{k})| &\geq \frac{\frac{3}{4}k_{stat}^2 m^2}{\left(k_{stat}\sqrt{k^2 + m^2} + k_1\sqrt{k_{stat}^2 + m^2}\right)\sqrt{k_1^2 + m^2}\sqrt{k_{stat}^2 + m^2}} \\
&= \frac{3m^2}{4\left(\sqrt{k^2 + m^2} + k_1\sqrt{1 + \left(\frac{m}{k_{stat}}\right)^2}\right)\sqrt{k_1^2 + m^2}\sqrt{1 + \left(\frac{m}{k_{stat}}\right)^2}}.
\end{aligned}$$

As $k_{stat} \geq \frac{m}{\sqrt{3}}$ (see 51) it follows:

$$|\dot{g}(\mathbf{k})| \geq \frac{3m^2}{8\left(\sqrt{k^2 + m^2} + 2k_1\right)\sqrt{k_1^2 + m^2}} =: h_1.$$

For $k_1 \geq 2k_{stat}$, g' is positive. Therefore similar as before:

$$\begin{aligned}
|g'(\mathbf{k})| &= \frac{k}{\sqrt{k^2 + m^2}} + a - y \cos(\vartheta) \\
&\geq \frac{k}{\sqrt{k^2 + m^2}} - \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} + \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} + a - y \\
&= \frac{k}{\sqrt{k^2 + m^2}} - \frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} = \frac{k\sqrt{k_{stat}^2 + m^2} - k_{stat}\sqrt{k^2 + m^2}}{\sqrt{k^2 + m^2}\sqrt{k_{stat}^2 + m^2}} \\
&= \frac{k^2(k_{stat}^2 + m^2) - k_{stat}^2(k^2 + m^2)}{(k^2 + m^2)(k_{stat}^2 + m^2)} \geq \frac{\frac{1}{4}k^2 m^2}{(k^2 + m^2)^2} =: h_2(\mathbf{k}).
\end{aligned}$$

Note, that h_1 and h_2 do not depend on a and \mathbf{y} .

We can use this estimate in (70). As g'' and \ddot{g} have uniform bounds in a and \mathbf{y} we get uniform estimates for I_2^1 and I_2^2 :

$$\begin{aligned}
\| I_2^1 \|_s &\leq \mu^{-2} \int_{k_1 < \frac{k_{stat}}{2}} \left\| 3 \frac{\ddot{\chi}_2}{h_1^2} + 3 \frac{\chi_2 \ddot{g}^2}{h_1^4} + 3 \frac{\chi_2 \ddot{g}}{h_1^3} \right\|_s d^3 k \\
&\leq \mu^{-2} \int_{\mathbb{R}^3} \left\| 3 \frac{\ddot{\chi}_2}{h_1^2} + 3 \frac{\tilde{\chi}_2 \ddot{g}^2}{h_1^4} + 3 \frac{\tilde{\chi}_2 \ddot{g}}{h_1^3} \right\|_s d^3 k \\
\| I_2^2 \|_s &\leq \mu^{-2} \int_{k_1 > 2k_{stat}} \left\| 3 \frac{\chi_2''}{h_2^2} + 3 \frac{\chi_2 g''^2}{h_2^4} + 3 \frac{\chi_2' g''}{h_2^3} \right\|_s d^3 k \\
&\leq \mu^{-2} \int_{k_1 \geq \frac{m}{\sqrt{3}}} \left\| 3 \frac{\tilde{\chi}_2''}{h_2^2} + 3 \frac{\tilde{\chi}_2 g''^2}{h_2^4} + 3 \frac{\tilde{\chi}_2' g''}{h_2^3} \right\|_s d^3 k .
\end{aligned}$$

Hence

$$\| I_2^1 \|_s + \| I_2^2 \|_s \leq \mu^{-2} C$$

with a constant C uniform in k_{stat} .

The integrals I_2^3 and I_2^4 can be estimated in a similar way, partial integration now be done with k_2 and k_3

$$| \partial_{k_j} g(\mathbf{k}) | = \frac{1}{\sqrt{k^2 + m^2}} + \frac{ak_j}{k^2} \leq \frac{1}{\sqrt{k^2 + m^2}} + \frac{a}{k} \quad \text{for } j = 1; 2$$

which is uniformly bounded away from zero on the area of integration.

So we have a uniform constant C with:

$$I_2 \leq \mu^{-2} C ,$$

and the lemma is proven for $y \in [a + \frac{1}{2}, a + 1]$.

(II) Next we prove the Lemma for $y < a + 1/2$.

We again have to assure, that all estimates are uniform in a and \mathbf{y} . In the last section the main difficulty we had to solve was, that g' near the stationary point was increasing with k_{stat}

(recall that $\lim_{y \rightarrow a+1} k_{stat} = \infty$).

So on the first view it seems to be simple to have uniform estimates for $y < a + 1/2$ just by setting $Q = \mathbb{R}^3$. But we have to face a new problem, which is, that the stationary point may be very close to zero. This is problematical in the differentiation of k appearing in our estimates.

For $a = 0$ this problem does not appear and the lemma is also proven for $y < \frac{1}{2}$ with $a = 0$.

As the divergence only appears for small k_{stat} we can set $k_{stat} < \frac{1}{2}$ (For $k_{stat} \geq \frac{1}{2}$ the estimates can be done very closely to the ones of (I), setting $Q = \mathbb{R}^3$).

We solve the problem by first "cutting out" the stationary point. We split our integral:

$$I = \int_{B(0, \sqrt{k_{stat}})} e^{-i\mu g(\mathbf{k})} \chi(\mathbf{k}) d^3 k + \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} e^{-i\mu g(\mathbf{k})} \chi(\mathbf{k}) d^3 k =: I_1 + I_2 .$$

As $k_{stat} < \frac{1}{2}$, the stationary point is inside the ball.

We estimate I_1 , writing it in spherical coordinates "centered" around the stationary point, by one partial integration:

$$\begin{aligned} \| I_1 \|_s &\leq \left\| \int_{B(0, \sqrt{k_{stat}})} e^{-i\mu g(\mathbf{k}')} \chi(\mathbf{k}') k'^2 dk' d\Omega \right\|_s \\ &\leq \left\| \mu^{-1} \int_{B(0, \sqrt{k_{stat}})} \left(\frac{\chi' k'^2}{g'} + \frac{2\chi k'}{g'} + \frac{\chi g'' k'^2}{g'^2} \right) dk' d\Omega \right\|_s . \end{aligned}$$

As $\chi \in \mathcal{G}$ all these terms are bounded, we have

$$\| I_1 \|_s \leq M \mu^{-1} \sqrt{k_{stat}}$$

We now estimate I_2 .

The first idea is to estimate this integral by two partial integrations. But the integrand still comes "very close" to the stationary point, where $(g')^{-1}$ is not bounded. So this procedure will not yield uniform bound in a and \mathbf{y} .

The trick to get uniform bound is to redo the split (60), (61) of (I) into the integral for $a + \frac{1}{2} < y < a + 1$.

$$I_2 = I_1^1 + I_1^2 + \partial_s I(s) |_{s=\xi}$$

now using $k = 0$ as the center for our Taylor-expansion. So we get:

$$\begin{aligned} I_1^1 &= \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} e^{i(g'(0)k + \frac{1}{2}g''(0)k^2)} \chi(0) d^3k \\ I_1^2 &= \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} e^{i(g'(0)k + \frac{1}{2}g''(0)k^2)} (\chi(\mathbf{k}) - \chi(0)) k^2 dk d\Omega \\ \partial_s I(s) |_{s=\xi} &= \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \lambda k^3 \tilde{f}(\mathbf{k}) e^{g_\xi(\mathbf{k})} \chi(k) k^2 dk d\Omega \end{aligned}$$

where

$$\begin{aligned} g'(0) &= \frac{k}{\sqrt{k^2 + m^2}} + a - y \cos(\vartheta) |_{k=0} = a - y \cos(\vartheta) \\ g''(0) &= \partial_k^2 g = \frac{m^2}{\sqrt{k^2 + m^2}^3} |_{k=0} = \frac{1}{m} \\ \tilde{f}(\mathbf{k}) &= (g(\mathbf{k}) - g(0) - g'(0)k - \frac{1}{2}g''(0)k^2) k^{-3} \\ g_\xi(\mathbf{k}) &= g(0) + g'(0)k + \frac{1}{2}g''(0)k^2 + \xi \tilde{f}(\mathbf{k}). \end{aligned} \tag{71}$$

As by similar argument concerning (67) g_ξ has only one stationary point $\tilde{\mathbf{k}}_{stat}$. One can easily see, that

$$g'(0) + g''(0)k = a - y \cos(\vartheta) + \frac{k}{m} \geq a - y \cos(\vartheta) + \frac{k}{\sqrt{k^2 + m^2}} = g'(\mathbf{k}).$$

Furthermore we have, that:

$$g'_\xi(\mathbf{k}) = (1 - \xi)(g'(0) + g''(0)k) + \xi g'(\mathbf{k}).$$

It follows, that

$$g'(0) + g''(0)k \geq g'_\xi \geq g'.$$

Therefore at $\mathbf{k} = \tilde{\mathbf{k}}_{stat}$ (where by definition $g'_\xi(\tilde{\mathbf{k}}_{stat}) = 0$) the g' has to be negative. It follows (recalling, that g' increases monotonously on the k_1 -axis), that

$$0 \leq \tilde{k}_{stat} \leq k_{stat} .$$

For the same reasons we have the zero point $\bar{\mathbf{k}}_{stat}$ of $g'(0) + g''(0)k$ (i.e. $\bar{\mathbf{k}}_{stat} = -\frac{g'(0)}{g''(0)}$):

$$0 \leq \bar{k}_{stat} \leq k_{stat} .$$

As the second derivative of $g''_\xi(\tilde{k}_{stat})$ is not equal to zero, we can define a function \tilde{g}_ξ with:

$$0 < M \leq \tilde{g}_\xi := |\mathbf{k} - \tilde{\mathbf{k}}_{stat}|^{-1} g_\xi . \quad (72)$$

The integral I_1^1 includes the leading term. It can be estimated like (62). The other terms can be estimated again by partial integrations. For that we define:

$$\zeta_1 := (\chi(\mathbf{k}) - \chi(0))k^2 =: \tilde{\zeta}_1 k^3 \quad \zeta_2 := \tilde{f}(\mathbf{k})\chi(k)k^5 =: \tilde{\zeta}_2 k^5$$

where $\tilde{\zeta}_{1,2}$ are bounded C^∞ -functions.

We now make two partial integrations in I_1^2 and three partial integrations in $\partial_s I(s)$ to get the estimates

$$\begin{aligned} \| I_1^2 \|_s &\leq \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k \left(\frac{1}{g'(0) + g''(0)k} \partial_k \left(\frac{\zeta_1}{g'(0) + g''(0)k} \right) \right) dk d\Omega \right\|_s \\ &= \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k \left(\frac{\zeta_1'}{(g'(0) + g''(0)k)^2} - \frac{\zeta_1 g''(0)}{(g'(0) + g''(0)k)^3} \right) dk d\Omega \right\|_s \\ \| \partial_s I(s) |_{s=\xi} \|_s &\leq \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k \left(\frac{1}{g'_\xi} \partial_k \left(\frac{1}{g'_\xi} \partial_k \left(\frac{\zeta_2}{g'_\xi} \right) \right) \right) dk d\Omega \right\|_s \\ &= \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k \left(\frac{1}{g'_\xi} \partial_k \left(\frac{\zeta_2'}{g_\xi'^2} - \frac{\zeta_2 g_\xi''}{g_\xi'^3} \right) \right) dk d\Omega \right\|_s \\ &= \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k \left(\frac{\zeta_2''}{g_\xi'^3} - 3 \frac{\zeta_2' g_\xi''}{g_\xi'^4} - \frac{\zeta_2 g_\xi'''}{g_\xi'^3} + 3 \frac{\zeta_2 g_\xi''^2}{g_\xi'^5} \right) dk d\Omega \right\|_s . \end{aligned}$$

So we can define functions f_j , $j = 1; \dots; 5$ which are bounded, with:

$$\begin{aligned} \| I_1^2 \|_s &\leq \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k (f_1 q_1^2 + f_2 q_1^3) dk d\Omega \right\|_s \\ \| \partial_s I(s) |_{s=\xi} \|_s &\leq \mu^{-2} \left\| \int_{\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})} \partial_k (f_3 q_2^3 + f_4 q_2^4 + f_5 q_2^5) dk d\Omega \right\|_s \end{aligned}$$

where

$$q_1 := \frac{k}{|\mathbf{k} - \bar{\mathbf{k}}_{stat}|} \quad q_2 := \frac{k}{|\mathbf{k} - \tilde{\mathbf{k}}_{stat}|}.$$

So it is only left to show, that $\partial_k q_1$ and $\partial_k q_2$ are bounded on $\mathbb{R}^3 \setminus B(0, \sqrt{k_{stat}})$. But this is easy:

$$\partial_k q_1 = \partial_k \frac{1}{\sqrt{1 - 2 \frac{k_{stat} \cos(\vartheta)}{k} + \frac{k_{stat}^2}{k^2}}} = \frac{1}{\sqrt{1 - 2 \frac{k_{stat} \cos(\vartheta)}{k} + \frac{k_{stat}^2}{k^2}}} \left(\frac{k_{stat}^2}{k^3} - \frac{k_{stat} \cos(\vartheta)}{k^2} \right)$$

for $k \geq \sqrt{k_{stat}}$ this term has obviously uniform bound.

The derivative of q_2 can be estimated in the same way. We only have to replace \tilde{k}_{stat} by \bar{k}_{stat} .

(III) For $y > a + 1$ we have no stationary point any more. So two partial integrations are possible without any problem. We again choose k_1 parallel to \mathbf{y}

$$\begin{aligned} \| I_2 \|_s &\leq \mu^{-2} \int \left\| \partial_k \left(\frac{1}{g'(\mathbf{k})} \partial_k \frac{\chi_2(\mathbf{k})}{g'(\mathbf{k})} \right) \right\|_s d^3 k = \mu^{-2} \int \left\| \partial_k \left(\frac{\chi'}{g'^2} - \frac{\chi g''}{g'^3} \right) \right\|_s d^3 k \\ &= \mu^{-2} \int \left\| \frac{\chi''}{g'^2} - 2 \frac{\chi' g''}{g'^3} - \frac{\chi' g''}{g'^3} - \frac{\chi g'''}{g'^3} + 3 \frac{\chi g''^2}{g'^4} \right\|_s d^3 k \end{aligned}$$

(f' means $\partial_{k_1} f$).

This integral still depends on k_{stat} . To get an estimate uniform in k_{stat} we use:

$$|g'(\mathbf{k})| = y - \frac{k_1}{\sqrt{k^2 + m^2}} - a \frac{k_1}{k} \geq 1 - \frac{k_1}{\sqrt{k^2 + m^2}} =: h(\mathbf{k}).$$

It follows:

$$\|I_2\|_s \leq \mu^{-2} \int \left\| \frac{\chi''}{h^2} + 3 \frac{\chi' g''}{h^3} + 3 \frac{\chi g''^2}{h^4} + \frac{\chi g'''}{h^3} \right\|_s d^3 k =: \mu^{-2} C.$$

4.2 Proof of equation(20)

For each \mathbf{k} we have two eigenstates for electrons. These two eigenstates span the two dimensional spinor subspace for electrons. In the standard representation of the Dirac matrices these two spinors² are:

$$s_{\mathbf{k}}^1 = \begin{pmatrix} \widehat{E}_k \\ 0 \\ k_1 \\ k^+ \end{pmatrix} \quad s_{\mathbf{k}}^2 = \begin{pmatrix} 0 \\ \widehat{E}_k \\ k^- \\ -k_1 \end{pmatrix}$$

where

$$k^\pm = k_2 \pm ik_3 \quad \widehat{E}_k = E_k + m \quad E_k = \sqrt{k^2 + m^2}.$$

If we now take any linear combination of these spinors $s_{\mathbf{k}} = a_{\mathbf{k}} s_{\mathbf{k}}^1 + b(\mathbf{k}) s_{\mathbf{k}}^2$ and compute for example $\langle s_{\mathbf{k}}^*, \alpha_1 s_{\mathbf{k}} \rangle$, we get (see(2)):

$$\begin{aligned} \langle s_{\mathbf{k}}^*, \alpha_1 s_{\mathbf{k}} \rangle &= \langle (a^*(\mathbf{k}) s_{\mathbf{k}}^{1*} + b^*(\mathbf{k}) s_{\mathbf{k}}^{2*}), \alpha_1 (a(\mathbf{k}) s_{\mathbf{k}}^1 + b(\mathbf{k}) s_{\mathbf{k}}^2) \rangle \\ &= \left(a^*(\mathbf{k}) \begin{pmatrix} \widehat{E}_k \\ 0 \\ k_1 \\ k^- \end{pmatrix} + b^*(\mathbf{k}) \begin{pmatrix} 0 \\ \widehat{E}_k \\ k^+ \\ -k_1 \end{pmatrix} \right), \left(a(\mathbf{k}) \begin{pmatrix} k_1 \\ -k^+ \\ \widehat{E}_k \\ 0 \end{pmatrix} + b(\mathbf{k}) \begin{pmatrix} k^- \\ k_1 \\ 0 \\ -\widehat{E}_k \end{pmatrix} \right) \\ &= (a^2(\mathbf{k}) + b^2(\mathbf{k})) 2\widehat{E}_k k_1. \end{aligned}$$

²The spinors here are not normalized!

With the normalization factor

$$\begin{aligned}\langle s_{\mathbf{k}}^*, s_{\mathbf{k}} \rangle &= (a^2(\mathbf{k}) + b^2(\mathbf{k}))(\widehat{E}_k^2 + k^2) = (a^2(\mathbf{k}) + b^2(\mathbf{k}))(E_k^2 + 2E_k m + m^2 + k^2) \\ &= (a^2(\mathbf{k}) + b^2(\mathbf{k}))(2E_k(E_k + m)) = (a^2(\mathbf{k}) + b^2(\mathbf{k}))(2E_k \widehat{E}_k)\end{aligned}$$

we get:

$$\langle s_{\mathbf{k}}^*, \alpha_1 s_{\mathbf{k}} \rangle = \frac{k_1}{\sqrt{k^2 + m^2}} \langle s_{\mathbf{k}}^*, s_{\mathbf{k}} \rangle .$$

Analogously we get:

$$\langle s_{\mathbf{k}}^*, \boldsymbol{\alpha} s_{\mathbf{k}} \rangle = \frac{\mathbf{k}}{\sqrt{k^2 + m^2}} \langle s_{\mathbf{k}}^*, s_{\mathbf{k}} \rangle .$$

By linearity (20) follows.

4.3 Proof of Lemma 3.4

(a)

To begin with, we consider the integral

$$I(\mathbf{x}) = \int \frac{1}{|\mathbf{x} - \mathbf{x}'|^j} f(\mathbf{x}') d^3 x' \tag{73}$$

for bounded, integrable $\|f\|_s$ and $j=1;2$.

For $j=1$ it has been proven by Ikebe [11], that I is Hölder continuous. We extend this to $j=2$. Therefore we need to estimate:

$$I(\mathbf{x} + \mathbf{h}) - I(\mathbf{x} - \mathbf{h})$$

for arbitrary \mathbf{h} with $h \leq \frac{1}{4}$ (We do not need to focus on $h > \frac{1}{4}$, as $I(\mathbf{x})$ is bounded). We split the integral into:

$$I(\mathbf{x} + \mathbf{h}) - I(\mathbf{x} - \mathbf{h}) = \tag{74}$$

$$\begin{aligned} &= \int_{B(\mathbf{x}, \sqrt{h})} \left(\frac{1}{|\mathbf{x} + \mathbf{h} - \mathbf{x}'|^2} - \frac{1}{|\mathbf{x} - \mathbf{h} - \mathbf{x}'|^2} \right) f(\mathbf{x}') d^3 x' \\ &+ \int_{B(\mathbf{x}, 1) \setminus B(\mathbf{x}, \sqrt{h})} \left(\frac{1}{|\mathbf{x} + \mathbf{h} - \mathbf{x}'|^2} - \frac{1}{|\mathbf{x} - \mathbf{h} - \mathbf{x}'|^2} \right) f(\mathbf{x}') d^3 x' \\ &+ \int_{\mathbb{R}^3 \setminus B(\mathbf{x}, 1)} \left(\frac{1}{|\mathbf{x} + \mathbf{h} - \mathbf{x}'|^2} - \frac{1}{|\mathbf{x} - \mathbf{h} - \mathbf{x}'|^2} \right) f(\mathbf{x}') d^3 x' =: I_1 + I_2 + I_3. \end{aligned} \tag{75}$$

For I_1 we have:

$$\| I_1 \|_s \leq 2 \sup_{x \in \mathbb{R}^3} \{ \| f(\mathbf{x}) \|_s \} \int_{B(\mathbf{x}, \sqrt{h})} \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} d^3 x'.$$

So we can find a constant $M < \infty$, so that

$$\| I_1(\mathbf{x}, \mathbf{h}) \|_s \leq M \sqrt{h} \quad \forall \mathbf{h} \in \mathbb{R}^3. \tag{76}$$

For I_2 we have, using $|\sqrt{h} - h| \leq \frac{1}{2}\sqrt{h}$:

$$\begin{aligned} \| I_2 \|_s &= \left\| \int_{B(\mathbf{x}, 1) \setminus B(0, \sqrt{h})} \left(\frac{1}{|\mathbf{x}' + \mathbf{h}|^2} - \frac{1}{|\mathbf{x}' - \mathbf{h}|^2} \right) f(\mathbf{x} - \mathbf{x}') d^3 x' \right\|_s \\ &\leq \sup_{x \in \mathbb{R}^3} \{ \| f(\mathbf{x}) \|_s \} \int_{B(\mathbf{x}, 1) \setminus B(0, \sqrt{h})} \frac{||\mathbf{x}' - \mathbf{h}|^2 - |\mathbf{x}' + \mathbf{h}|^2|}{|\mathbf{x}' + \mathbf{h}|^2 |\mathbf{x}' - \mathbf{h}|^2} d^3 x' \\ &\leq \sup_{x \in \mathbb{R}^3} \{ \| f(\mathbf{x}) \|_s \} \int_{B(\mathbf{x}, 1) \setminus B(0, \sqrt{h})} \frac{4h x'}{|\mathbf{x}' + \mathbf{h}|^2 |\mathbf{x} - \mathbf{h}|^2} d^3 x' \\ &\leq \sup_{x \in \mathbb{R}^3} \{ \| f(\mathbf{x}) \|_s \} \int_{B(\mathbf{x}, 1) \setminus B(0, \sqrt{h})} \frac{8h}{x'^3} d^3 x'. \end{aligned}$$

So we can find a constant $M < \infty$, so that

$$\| I_2(\mathbf{x}, \mathbf{h}) \|_s \leq M \sqrt{h} \quad \forall \mathbf{h} \in \mathbb{R}^3. \tag{77}$$

For I_3 we have, using similar reasoning as above:

$$\| I_3 \|_s \leq \int_{\mathbb{R}^3 \setminus B(\mathbf{0},1)} \frac{8h}{x'^3} \| f(\mathbf{x} - \mathbf{x}') \|_s d^3 x' \leq 8h \int \| f(\mathbf{x} - \mathbf{x}') \|_s d^3 x' .$$

Since f is absolutely integrable, we can find a constant $M < \infty$, so that

$$\| I_3(\mathbf{x}, \mathbf{h}) \|_s \leq Mh \quad \forall \mathbf{h} \in \mathbb{R}^3 . \quad (78)$$

We use this estimate on (36), observing, that $G_{\mathbf{k}}^+$ multiplied by $A\tilde{\varphi}_{\mathbf{k}}^s$ is essentially of the form of the integrals in (73). Therefore:

$$\| \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x} + \mathbf{h}) - \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) \|_s \leq M\sqrt{h} \quad \forall \mathbf{h} \in \mathbb{R}^3 . \quad (79)$$

Now we want to focus on integrals of the form (73) for $j=2$ where $f(\mathbf{x})$ satisfies:

$$\| f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \|_s \leq M\sqrt{h} . \quad (80)$$

We do a similar splitting as in (74). Now we have for I_1 , using (80):

$$\begin{aligned} \| I_1 \|_s &\leq \left\| \int_{B(\mathbf{x}, \sqrt{h})} \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} (f(\mathbf{x}' + \mathbf{h}) - f(\mathbf{x}' - \mathbf{h})) d^3 x' \right\|_s \\ &\leq \left| \int_{B(\mathbf{x}, \sqrt{h})} \frac{1}{|\mathbf{x} - \mathbf{x}'|^2} M\sqrt{h} d^3 x' \right| . \end{aligned}$$

Thus with an appropriate $\tilde{M} < \infty$:

$$\| I_1(\mathbf{x}, \mathbf{h}) \|_s \leq Mh \quad \forall \mathbf{h} \in \mathbb{R}^3 . \quad (81)$$

For I^2 we have:

$$\begin{aligned}
\| I_2^2 \|_s &= \left\| \int_{B(0,1) \setminus B(0,\sqrt{h})} \left(\frac{1}{|\mathbf{x}' + \mathbf{h}|^2} - \frac{1}{|\mathbf{x}' - \mathbf{h}|^2} \right) f(\mathbf{x} - \mathbf{x}') d^3 x' \right\|_s \\
&= \left\| \int_{B(0,1) \setminus B(0,\sqrt{h})} \frac{|\mathbf{x}' - \mathbf{h}|^2 - |\mathbf{x}' + \mathbf{h}|^2}{|\mathbf{x}' - \mathbf{h}|^2 |\mathbf{x}' + \mathbf{h}|^2} f(\mathbf{x} - \mathbf{x}') d^3 x' \right\|_s .
\end{aligned}$$

Since the fraction under this integral is point-symmetric to zero, we can estimate the integral by:

$$\begin{aligned}
\| I_2^2 \|_s &\leq \left\| \int_{B(0,1) \setminus B(0,\sqrt{h})} \left| \frac{|\mathbf{x}' - \mathbf{h}|^2 - |\mathbf{x}' + \mathbf{h}|^2}{|\mathbf{x}' - \mathbf{h}|^2 |\mathbf{x}' + \mathbf{h}|^2} \right| (f(\mathbf{x} - \mathbf{x}') - f(\mathbf{x} + \mathbf{x}')) d^3 x' \right\|_s \\
&\leq \left\| \int_{B(0,1) \setminus B(0,\sqrt{h})} \left| \frac{|\mathbf{x}' - \mathbf{h}|^2 - |\mathbf{x}' + \mathbf{h}|^2}{|\mathbf{x}' - \mathbf{h}|^2 |\mathbf{x}' + \mathbf{h}|^2} \right| M \sqrt{2x'} d^3 x' \right\|_s \\
&\leq \left\| \int_{B(0,1) \setminus B(0,\sqrt{h})} 4 \left| \frac{2h}{x'^3} \right| M \sqrt{2x'} d^3 x' \right\|_s \leq 16\pi M \sqrt{2} \int_{\sqrt{h}}^1 \left| \frac{2h}{x'^{\frac{5}{2}}} \right| d^3 x' \|_s .
\end{aligned}$$

So we can find a $\widetilde{M} < \infty$ with:

$$\| I_2^2(\mathbf{x}, \mathbf{h}) \|_s \leq \widetilde{M} h \quad \forall \mathbf{h} \in \mathbb{R}^3 . \quad (82)$$

For I_3 we do the same estimations as before.

Applying this to (36) we obtain the Hölder continuity of degree 1 for $\widetilde{\varphi}_{\mathbf{k}}^s$.

(b)

Assume that $\widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ satisfies (36) and is Hölder continuous of degree 1. Inserting $\widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ in the right hand side of (32) leads to:

$$\begin{aligned}
H \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) &= (H_0 + \mathcal{A}(\mathbf{x})) \left(\varphi_{\mathbf{k}}^s(\mathbf{x}) - \int \mathcal{A}(\mathbf{x}') G_k^+(\mathbf{x} - \mathbf{x}') \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \right) \\
&= (E_k + \mathcal{A}(\mathbf{x})) \varphi_{\mathbf{k}}^s(\mathbf{x}) - (H_0 + \mathcal{A}(\mathbf{x})) \int \mathcal{A}(\mathbf{x}') G_k^+(\mathbf{x} - \mathbf{x}') \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}') d^3 x' .
\end{aligned}$$

For (32) this term has to be equal to $E_k \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$. So we have to prove, that

$$(H_0 - E_k) \int G_k^+(\mathbf{x} - \mathbf{x}') A(\mathbf{x}') \tilde{\varphi}_k^s(\mathbf{x}') d^3 x' = A(\mathbf{x}) \tilde{\varphi}_k^s(\mathbf{x}).$$

In other words we have to prove, that with f Hölder continuous of degree 1:

$$(H_0 - E_k) \int G_k^+(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' = f(\mathbf{x}). \quad (83)$$

G_k^+ can be written as [14]:

$$G_k^+(\mathbf{x}) = (H_0 + E_k) \frac{e^{ikx}}{4\pi x} =: (H_0 + E_k) G_k^{KG}$$

with

$$(H_0 - E_k)(H_0 + E_k) G_k^{KG} = (\Delta - k^2) G_k^{KG} = \delta. \quad (84)$$

So for (83) we need to show, that:

$$(H_0 - E_k)(H_0 + E_k) \int G_k^{KG}(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' = (\Delta - k^2) \int G_k^{KG}(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' = f(\mathbf{x}). \quad (85)$$

We define for $\varepsilon > 0$ the following function G_k^ε :

$$G_k^\varepsilon(\mathbf{x}) := G_k^{KG}(\mathbf{x}) \text{ for } x \geq \varepsilon \quad G_k^\varepsilon(\mathbf{x}) = G_k^{KG}(\mathbf{x})(1 - e^{-\frac{x}{\varepsilon}}) \text{ for } x < \varepsilon. \quad (86)$$

We denote

$$G'_k(\mathbf{x}) = \nabla G_k^{KG} = \frac{ik\mathbf{x}e^{ikx}}{4\pi x^2} + \frac{\mathbf{x}e^{ikx}}{x^3}. \quad (87)$$

We split the right hand side of (85) into:

$$(\Delta - k^2) \int (G_k^{KG}(\mathbf{x} - \mathbf{x}') - G_k^\varepsilon(\mathbf{x}')) f(\mathbf{x}') d^3 x' + (\Delta - k^2) \int G_k^\varepsilon(\mathbf{x}') f(\mathbf{x}') d^3 x'. \quad (88)$$

By definition of G_k^{KG} (84) we have outside the Ball $B(0, \varepsilon)$:

$$(\Delta - k^2)G_k^\varepsilon(\mathbf{x}) = (\Delta - k^2)G_k^{KG}(\mathbf{x}) = 0 . \quad (89)$$

So for the first summand we have:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left\| (\Delta - k^2) \int (G_k^{KG}(\mathbf{x} - \mathbf{x}') - G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) f(\mathbf{x}') d^3 x' \right\|_s \\ &= \lim_{\varepsilon \rightarrow 0} \left\| \Delta \int_{B(\mathbf{x}, \varepsilon)} (G_k^{KG}(\mathbf{x} - \mathbf{x}') - G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) f(\mathbf{x}') d^3 x' \right\|_s \\ &= \lim_{\varepsilon \rightarrow 0} \left\| \nabla \int_{B(\mathbf{x}, \varepsilon)} \nabla (G_k^{KG}(\mathbf{x} - \mathbf{x}') - G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) f(\mathbf{x}') d^3 x' \right\|_s \\ &= \lim_{\varepsilon \rightarrow 0} \left\| \nabla \int_{B(\mathbf{x}, \varepsilon)} (\nabla_{x'} (G_k^{KG}(\mathbf{x} - \mathbf{x}') - G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) f(\mathbf{x}') d^3 x' \right\|_s \\ &= \lim_{\varepsilon \rightarrow 0} \left\| \nabla \int_{B(\mathbf{x}, \varepsilon)} (\nabla_{x'} (G_k^{KG}(\mathbf{x}') - G_k^\varepsilon(\mathbf{x}')) f(\mathbf{x} - \mathbf{x}') d^3 x' \right\|_s \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} \left\| (\nabla_{x'} (G_k^{KG}(\mathbf{x}') - G_k^\varepsilon(\mathbf{x}')) \frac{f(\mathbf{x} - \mathbf{x}') - f(\mathbf{x} + \mathbf{h} - \mathbf{x}')}{h} \right\|_s d^3 x' . \end{aligned}$$

As f is Hölder continuous, the last term can be estimated by:

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \left\| (\Delta - k^2) \int (G_k^{KG}(\mathbf{x} - \mathbf{x}') - G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) f(\mathbf{x}') d^3 x' \right\|_s \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_{B(0, \varepsilon)} \left| \nabla_{x'} (G_k^{KG}(\mathbf{x}') - G_k^\varepsilon(\mathbf{x}')) M \right| d^3 x' \\ &\leq \lim_{\varepsilon \rightarrow 0} \int_{B(0, \varepsilon)} \left| \left(G_k'(\mathbf{x}') - G_k'(\mathbf{x}')(1 - e^{-\frac{x^2}{\varepsilon - x}}) - G_k^\varepsilon(\mathbf{x}') \frac{-\varepsilon}{(x - \varepsilon)^2} \right) M \right| d^3 x' = 0 . \end{aligned}$$

For the second summand, we use (89) and the mean value theorem

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} (\Delta - k^2) \int G_k^\varepsilon(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} (\Delta - k^2) \int_{B(\mathbf{x}, \varepsilon)} G_k^\varepsilon(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} (\Delta - k^2) G_k^\varepsilon(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} (\Delta - k^2) (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}') e^{ik|\mathbf{x}-\mathbf{x}'|}) f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} (\Delta - k^2) (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}') e^{ik|\mathbf{x}-\mathbf{x}'|}) f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} (\Delta + 2ik\nabla) (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) e^{ik|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} \Delta (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) e^{ik|\mathbf{x}-\mathbf{x}'|} f(\mathbf{x}') d^3 x' \\
&= \lim_{\varepsilon \rightarrow 0} e^{ik|\mathbf{x}-\mathbf{x}_\varepsilon|} f(\mathbf{x}_\varepsilon) \int_{B(\mathbf{x}, \varepsilon)} \Delta (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) d^3 x'
\end{aligned}$$

where $\mathbf{x}_\varepsilon \in B(\mathbf{x}, \varepsilon)$ using the positivity of

$$\Delta (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) = 2 \frac{1 - e^{\frac{x}{\varepsilon-x}}}{4\pi x^3} + 2 \frac{\varepsilon e^{\frac{x}{\varepsilon-x}}}{(x-\varepsilon)^2 4\pi x^2} + \frac{(\varepsilon^2 + 2x\varepsilon) e^{\frac{x}{\varepsilon-x}}}{(x-\varepsilon)^4 4\pi x} \geq 0.$$

Hence with Gauss' theorem and (87)

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} (\Delta - k^2) \int G_k^\varepsilon(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x' \\
&= f(\mathbf{x}) \lim_{\varepsilon \rightarrow 0} \int_{B(\mathbf{x}, \varepsilon)} \Delta (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) d^3 x' \\
&= f(\mathbf{x}) \lim_{\varepsilon \rightarrow 0} \int_{\partial B(\mathbf{x}, \varepsilon)} \nabla (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^\varepsilon(\mathbf{x} - \mathbf{x}')) \cdot \mathbf{n} d\Omega \\
&= f(\mathbf{x}) \lim_{\varepsilon \rightarrow 0} \int_{\partial B(\mathbf{x}, \varepsilon)} \nabla (e^{-ik|\mathbf{x}-\mathbf{x}'|} G_k^{KG}(\mathbf{x} - \mathbf{x}')) \cdot \mathbf{n} |\mathbf{x} - \mathbf{x}'|^2 d\Omega \\
&= f(\mathbf{x}) \lim_{\varepsilon \rightarrow 0} \int_{\partial B(\mathbf{x}, \varepsilon)} \frac{\mathbf{x} - \mathbf{x}'}{4\pi |\mathbf{x} - \mathbf{x}'|^3} \cdot \mathbf{n} |\mathbf{x} - \mathbf{x}'|^2 d\Omega = f(\mathbf{x})
\end{aligned}$$

and (b) is proved.

We show now, that for any $\mathbf{k} \in \mathbb{R}^3$ there exists a unique solution $\tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x})$ of (36).

Using the definition of the $\zeta_{\mathbf{k}}^s(\mathbf{x})$ (see 39) in (36) yields:

$$\zeta_{\mathbf{k}}^s(\mathbf{x}) = v_{\mathbf{k}}(\mathbf{x}) - \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \quad (90)$$

where

$$v_{\mathbf{k}}(\mathbf{x}) := - \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \varphi_{\mathbf{k}}^s(\mathbf{x}') d^3 x' . \quad (91)$$

It suffices to prove, that (90) has a unique solution for any $\mathbf{k} \in \mathbb{R}^3$. For the Schrödinger Greens-function, this has already been proven by Ikebe [11]. We want to proceed in the same way.

Let \mathcal{B} be the Banach space of all continuous functions tending uniformly to zero as $x \rightarrow \infty$. Due to (48) $v(\mathbf{x}) \in \mathcal{B}$. Ikebe uses the Riesz-Schauder theory of completely continuous operators in a Banach space [13]:

If T is a completely continuous operator in \mathcal{B} , then for any given $g \in \mathcal{B}$ the equation

$$f = g + Tf \quad (92)$$

has a unique solution in \mathcal{B} if $\tilde{f} = T\tilde{f}$ implies that $\tilde{f} = 0$.

Defining the integral operator T by:

$$Tf(\mathbf{x}) := - \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') d^3 x'$$

and using v for g , (92) is equivalent to (90). Note, that this operator is completely continuous by the proof of Lemma 3.4(a) following a similar argumentation as in [11] Lemma 4.2. So it is left to show, that the integral equation

$$\tilde{f}(\mathbf{x}) = - \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \tilde{f}(\mathbf{x}') d^3 x' \quad (93)$$

has the unique solution $\tilde{f} \equiv 0$.

Obviously $\tilde{f} \equiv 0$ is a solution of (93). By virtue of (48) any solution of (93) has to be of order x^{-1} . Furthermore \tilde{f} satisfies

$$(-\Delta - k^2 + \mathcal{A})\tilde{f} = 0 \quad (94)$$

which can be shown by direct calculation.

Following Ikebe, $\tilde{f} \equiv 0$ is the only solution of (93).

(c)

(c)i) follows directly from (48). For (c)ii) we need to work more. We exemplarily prove (c)ii) for $j=1,2$.

Heuristically deriving (90) with respect to k will yield $\partial_k \zeta$. We denote the function we get by this formal method by $\zeta_{\mathbf{k}}^{\prime s}$. Then

$$\zeta_{\mathbf{k}}^{\prime s}(\mathbf{x}) = \partial_k v_{\mathbf{k}}(\mathbf{x}) - \int \mathcal{A}(\mathbf{x}') \partial_k G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' - \int \mathcal{A}(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^{\prime s}(\mathbf{x}') d^3 x' . \quad (95)$$

We will now show, that this integral equation has a unique solution. We define

$$p(\mathbf{x}) := \partial_k v_{\mathbf{k}}(\mathbf{x}) + \int \mathcal{A}(\mathbf{x}') \partial_k G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \quad (96)$$

$$\bar{\zeta}_{\mathbf{k}}^s(\mathbf{x}) := \zeta_{\mathbf{k}}^{\prime s}(\mathbf{x}) - p(\mathbf{x}) \quad (97)$$

so $\bar{\zeta}_{\mathbf{k}}^s$ satisfies:

$$\bar{\zeta}_{\mathbf{k}}^s(\mathbf{x}) = - \int \mathcal{A}(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') p(\mathbf{x}') d^3 x' - \int \mathcal{A}(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \bar{\zeta}_{\mathbf{k}}^{\prime s}(\mathbf{x}') d^3 x' .$$

Since

$$v'(\mathbf{x}) := - \int \mathcal{A}(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') v'(\mathbf{x}') d^3 x' \in \mathcal{B}$$

this integral equation again has a unique solution, so does (96).

We will now show, that $\zeta' = \partial_k \zeta$.

We define the integral of ζ' :

$$\tilde{\zeta}_{k,\vartheta,\varphi}^s(\mathbf{x}) := \zeta_0^s(\mathbf{x}) + \int_0^k \zeta_{k',\vartheta,\varphi}^{t_s}(\mathbf{x}) dk'. \quad (98)$$

Obviously $\partial_k \tilde{\zeta}_{\mathbf{k}}^s = \zeta_{\mathbf{k}}^{s'}$ and $\tilde{\zeta}_0^s = \zeta_0^s$. Using (90) and (96) in (98) leads to:

$$\begin{aligned} \tilde{\zeta}_{\mathbf{k}}^s(\mathbf{x}) &= \zeta_0^s(\mathbf{x}) + \int_0^k \zeta_{k',\vartheta,\varphi}^{t_s}(\mathbf{x}) dk' \\ &= v_0(\mathbf{x}) - \int A(\mathbf{x}') G_0^+(\mathbf{x} - \mathbf{x}') \zeta_0^s(\mathbf{x}') d^3 x' + \int_0^k \partial_{k'} v_{\mathbf{k}'}(\mathbf{x}) dk' \\ &\quad - \int \int A(\mathbf{x}') \partial_{k'} G_{\mathbf{k}'}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}'}^s(\mathbf{x}') d^3 x' + \int A(\mathbf{x}') G_{\mathbf{k}'}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}'}^{t_s}(\mathbf{x}') d^3 x' dk' \\ &= v_{\mathbf{k}}(\mathbf{x}) - \int A(\mathbf{x}') G_0^+(\mathbf{x} - \mathbf{x}') \tilde{\zeta}_0^s(\mathbf{x}') d^3 x' \\ &\quad - \int \int A(\mathbf{x}') \partial_{k'} G_{\mathbf{k}'}^+(\mathbf{x} - \mathbf{x}') \tilde{\zeta}_{\mathbf{k}'}^s(\mathbf{x}') d^3 x' + \int A(\mathbf{x}') G_{\mathbf{k}'}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}'}^{t_s}(\mathbf{x}') d^3 x' dk' \\ &\quad - \int \int A(\mathbf{x}') \partial_{k'} (G_{\mathbf{k}'}^+(\mathbf{x} - \mathbf{x}') \tilde{\zeta}_{\mathbf{k}'}^s(\mathbf{x}')) d^3 x' dk' \\ &= v_{\mathbf{k}}(\mathbf{x}) - \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \tilde{\zeta}_{\mathbf{k}}^s(\mathbf{x}') d^3 x'. \end{aligned}$$

So $\tilde{\zeta}_{\mathbf{k}}^s$ satisfies (90). As the solution is unique, it follows, that $\tilde{\zeta}_{\mathbf{k}}^s = \zeta_{\mathbf{k}}^s$, hence

$$\partial_k \zeta_{\mathbf{k}}^s = \zeta_{\mathbf{k}}^{s'}.$$

By (48) $\tilde{\zeta}_{\mathbf{k}}^s$ and $p(\mathbf{x})$ have uniform bound, so

$$\sup_{\mathbf{x} \in \mathbb{R}^3} \|\partial_k \zeta_{\mathbf{k}}^s(\mathbf{x})\|_s < \infty.$$

For the second derivative we have:

$$\partial_k^2 \frac{\zeta^s}{x+1} = \partial_k \frac{\bar{\zeta}^s}{x+1} + \partial_k \frac{p}{x+1}.$$

The proof of the existence and uniqueness of $\partial_k \frac{\bar{\zeta}^s}{x}$ is the same as for $\partial_k \zeta^s$, furthermore $\partial_k \bar{\zeta}^s$ is

bounded uniformly in \mathbf{x} .

For $\partial_k \frac{p}{x+1}$ we have:

$$\begin{aligned}
\| \partial_k \frac{p}{x+1} \|_s &= \| \frac{1}{x+1} \partial_k (\partial_k v_{\mathbf{k}}(\mathbf{x}) - \int \mathcal{A}(\mathbf{x}') \partial_k G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x') \|_s \\
&= \| \frac{1}{x+1} (\partial_k^2 v_{\mathbf{k}}(\mathbf{x}) - \int \mathcal{A}(\mathbf{x}') \partial_k^2 G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \\
&\quad - \int \mathcal{A}(\mathbf{x}') \partial_k G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \partial_k \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x') \|_s . \tag{99}
\end{aligned}$$

Note, that

$$| \partial_k^2 G_{\mathbf{k}}^+(\mathbf{x}) | = | x^2 S_{\mathbf{k}}^+(\mathbf{x}) + x \partial_k S_{\mathbf{k}}^+(\mathbf{x}) + \partial_k^2 S_{\mathbf{k}}^+(\mathbf{x}) | \leq M(xk + \frac{k}{x^2}) . \tag{100}$$

Observing (100) and (35) $\frac{\partial_k^2 G_{\mathbf{k}}^+}{x+1}$ and $\partial_k \zeta_{\mathbf{k}}^s$ are bounded uniformly in \mathbf{x} , we have also, that

$$\frac{1}{x+1} (\partial_k^2 v_{\mathbf{k}}(\mathbf{x}) - \int \mathcal{A}(\mathbf{x}') \partial_k G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \partial_k \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x')$$

is uniformly bounded in \mathbf{x} .

For the other summand we have get:

$$\begin{aligned}
&\sup_{x, k \in \mathbb{R}^3} \| \frac{1}{x+1} \int \mathcal{A}(\mathbf{x}') \partial_k^2 G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \|_s \\
&\leq \sup_{x \in \mathbb{R}^3} \| \frac{1}{x+1} \int \mathcal{A}(\mathbf{x}') (\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \|_s \\
&\leq \sup_{x \in \mathbb{R}^3} \| \int \mathcal{A}(\mathbf{x}') \frac{M(\mathbf{x} - \mathbf{x}')}{(x+1)(x'+1)} (x'+1) \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \|_s \\
&\leq \sup_{x \in \mathbb{R}^3} \| \int \mathcal{A}(\mathbf{x}') M(x'+1) \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' \|_s < \infty
\end{aligned}$$

This proves (c)ii).

(c)iii)

The proof of (c)iii) is very similar to the proof of (c)ii). The only difference is, that we get

new functions $p(\mathbf{x})$.

$$p(\mathbf{x}) = k^{|\gamma|-1} D_{\mathbf{k}}^{\gamma} v_{\mathbf{k}}(\mathbf{x}) + \int A(\mathbf{x}') k^{|\gamma|-1} D_{\mathbf{k}}^{\gamma} G_{\mathbf{k}}^{+}(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' .$$

To have $p(\mathbf{x})$ in \mathcal{B} one only has to assure, that $k^{|\gamma|-1} D_{\mathbf{k}}^{\gamma} k$ is bounded for $|\gamma| \leq 2$, which follows by direct calculation.

(d)

For potentials satisfying Condition A (3) the scattering system (H, H_0) is asymptotically complete (see [14]), i.e. for any scattering state ψ there exists a free outgoing asymptotic ψ_{out} with:

$$\lim_{t \rightarrow \infty} \|\psi(\mathbf{x}, t) - \psi_{\text{out}}(\mathbf{x}, t)\| = 0 . \quad (101)$$

We write this, using the Fourier transform $\widehat{\psi}_{\text{out}}^s$ of ψ_{out} :

$$\lim_{t \rightarrow \infty} \|\psi(\mathbf{x}, t) - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) \varphi_{\mathbf{k}}^s(\mathbf{x}, t) d^3 k\| = 0 .$$

We shall show that

$$\lim_{t \rightarrow \infty} \|\int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) (\widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}, t) - \varphi_{\mathbf{k}}^s(\mathbf{x}, t)) d^3 k\| = 0 . \quad (102)$$

With that:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \|\psi(\mathbf{x}, t) - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}, t) d^3 k\| \\ &= \lim_{t \rightarrow \infty} \|e^{-iHt} (\psi(\mathbf{x}) - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) d^3 k)\| \\ &= \|\psi(\mathbf{x}) - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) d^3 k\| = 0 . \end{aligned}$$

which establishes (40). For (102) we consider

$$\begin{aligned}
& \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{out,s}(\mathbf{k}) (\widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}, t) - \varphi_{\mathbf{k}}^s(\mathbf{x}, t)) d^3 k \\
&= \int (2\pi)^{-\frac{3}{2}} e^{iE_{\mathbf{k}} t} \widehat{\psi}_{out,s}(\mathbf{k}) \zeta_{\mathbf{k}}^s(\mathbf{x}) d^3 k \\
&= \int (2\pi)^{-\frac{3}{2}} e^{iE_{\mathbf{k}} t} \widehat{\psi}_{out,s}(\mathbf{k}) v_{\mathbf{k}}^s(\mathbf{x}) d^3 k \\
&\quad - \int (2\pi)^{-\frac{3}{2}} e^{iE_{\mathbf{k}} t} \widehat{\psi}_{out,s}(\mathbf{k}) \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') \zeta_{\mathbf{k}}^s(\mathbf{x}') d^3 x' d^3 k \\
&=: \xi_1(\mathbf{x}) + \xi_2(\mathbf{x}) .
\end{aligned} \tag{103}$$

For the \mathbf{k} -integration of ξ_1 we introduce (91) and (35) and then use Lemma 3.1, setting:

$$\mu = t; \quad a = t^{-1} |\mathbf{x} - \mathbf{x}'|; \quad \mathbf{y} = t^{-1} \mathbf{x}'; \quad k' = k; \quad \chi(\mathbf{k}') = (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{out,s}(\mathbf{k}').$$

Furthermore we recall, that:

$$\begin{aligned}
\frac{k_{stat}}{\sqrt{k_{stat}^2 + m^2}} + a - y &= 0 \\
k_{stat}^2 &= (k_{stat}^2 + m^2)(y - a)^2 \\
k_{stat} &= m(y - a) \sqrt{k_{stat}^2 + m^2} = m \frac{x'}{t} \sqrt{k_{stat}^2 + m^2} .
\end{aligned}$$

For ξ_2 we set:

$$\mu = t; \quad a = t^{-1} |\mathbf{x} - \mathbf{x}'|; \quad \mathbf{y} = 0; \quad k' = k; \quad \chi(\mathbf{k}') = (2\pi)^{-\frac{3}{2}} \zeta_{\mathbf{k}'}^s(\mathbf{k}') \widehat{\psi}_{out,s}(\mathbf{k}').$$

Hence by (17) we obtain for (103) that there exists $M < \infty$ uniform in \mathbf{y} and a , such that:

$$\begin{aligned}
\| \xi_1(\mathbf{x}) + \xi_2(\mathbf{x}) \|_s &\leq M t^{-\frac{3}{2}} \left| \int A(\mathbf{x}') G_{\mathbf{k}}^+(\mathbf{x} - \mathbf{x}') (1 + x') d^3 x' \right| \\
&=: M t^{-\frac{3}{2}} G(\mathbf{x}) .
\end{aligned} \tag{104}$$

The integral $G(\mathbf{x})$ is bounded and goes to zero in the limit $x \rightarrow \infty$ (see 48). This we shall use

in the following estimate. For (102) we need to control

$$\lim_{t \rightarrow \infty} \|\xi_1 + \xi_2\| = \lim_{t \rightarrow \infty} \left(\int \|\xi_1 + \xi_2\|_s^2 d^3x \right)^{\frac{1}{2}} .$$

We split this integral into three parts, which are time dependent by introducing for all $\varepsilon > 0$:

$$\begin{aligned} \rho_\varepsilon(\mathbf{x}) &= \mathbb{I}_{B(0,\varepsilon t)} , \text{ the indicator function of the set } B(0,\varepsilon t) \\ \tilde{\rho}_\varepsilon(\mathbf{x}) &= \mathbb{I}_{B(0,t) \setminus B(0,\varepsilon t)} \\ \rho_{\text{out}}(\mathbf{x}) &= \mathbb{I}_{\mathbb{R}^3 \setminus B(0,t)} \end{aligned}$$

thus splitting our integral into:

$$\begin{aligned} \lim_{t \rightarrow \infty} \int \|\xi_1 + \xi_2\|_s^2 d^3x &= \lim_{t \rightarrow \infty} \int \rho_\varepsilon(\mathbf{x}) \|\xi_1 + \xi_2\|_s^2 d^3x \\ &\quad + \lim_{t \rightarrow \infty} \int \tilde{\rho}_\varepsilon(\mathbf{x}) \|\xi_1 + \xi_2\|_s^2 d^3x \\ &\quad + \lim_{t \rightarrow \infty} \int \rho_{\text{out}}(\mathbf{x}) \|\xi_1 + \xi_2\|_s^2 d^3x \\ &=: I_1 + I_2 + I_3 . \end{aligned} \tag{105}$$

The last part of this integral is the part, that lies outside the light cone. For large times, all wavefunctions which are solutions of the free or full Dirac equation will lie inside the light cone.

By virtue of (29):

$$\begin{aligned} \lim_{t \rightarrow \infty} \|\rho_{\text{out}}(\mathbf{x}) \psi_{\text{out},s}(\mathbf{x})\| &= 0 \\ \lim_{t \rightarrow \infty} \|\rho_{\text{out}}(\mathbf{x}) \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) \varphi_{\mathbf{k}}^s(\mathbf{x}) d^3k\| &= 0 \end{aligned}$$

or by (101):

$$\lim_{t \rightarrow \infty} \|\rho_{\text{out}}(\mathbf{x}) \int (2\pi)^{-\frac{3}{2}} \widehat{\psi}_{\text{out},s}(\mathbf{k}) \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) d^3k\| = 0 .$$

By (103) it follows, that:

$$I_3 = \lim_{t \rightarrow \infty} \left\| \rho_{\text{out}}(\mathbf{x}) \int \widehat{\psi}_{\text{out},s}(\mathbf{k}) (\widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) - \varphi_{\mathbf{k}}^s(\mathbf{x})) d^3k \right\| = 0.$$

Now we use (104) on:

$$I_1 \leq M^2 \lim_{t \rightarrow \infty} (\sup_{x \leq \varepsilon t} \{G(x)\})^2 t^{-3} \frac{4\pi}{3} (\varepsilon t)^3 = C\varepsilon^3.$$

Since ε is arbitrary, $I_1 = 0$.

For I_2 we have:

$$\begin{aligned} I_2 &= \lim_{t \rightarrow \infty} \left| \int \widetilde{\rho}_\varepsilon(\mathbf{x}) \|\xi_1 + \xi_2\|_s^2 d^3x \right| \\ &= \lim_{t \rightarrow \infty} \left| M^2 \int t^{-3} \widetilde{\rho}_\varepsilon(\mathbf{x}) G(\mathbf{x})^2 d^3x \right| \\ &\leq \lim_{t \rightarrow \infty} \sup_{x \geq \varepsilon t} |G(\mathbf{x})|^2 = 0 \end{aligned}$$

and (40) is proved.

We first prove (41) for wavefunctions, where ψ_{out} is in $L^1 \cap L^2$. The general result can then be obtained by density arguments.

Therefore we again use the unitarity of the time propagator:

$$\begin{aligned} \int (2\pi)^{-\frac{3}{2}} \langle \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}) \rangle d^3x &= \lim_{t \rightarrow \infty} \int (2\pi)^{-\frac{3}{2}} e^{iHt} \langle \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), e^{-iHt} \psi(\mathbf{x}) \rangle d^3x \\ &= \lim_{t \rightarrow \infty} e^{iEt} \int (2\pi)^{-\frac{3}{2}} \langle \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}, t) \rangle d^3x \\ &= \lim_{t \rightarrow \infty} e^{iEt} \int_{B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}, t) \rangle d^3x \\ &\quad + \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}, t) \rangle d^3x. \end{aligned}$$

By asymptotical completeness (101) we obtain therefore

$$\begin{aligned} \int (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}) \rangle d^3x &= \lim_{t \rightarrow \infty} e^{iEt} \int_{B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x \\ &+ \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x . \end{aligned}$$

By the free scattering into cones theorem, the first integral of the right hand side goes to zero because any freely evolving wavefunction leaves any bounded region. For the second integral we write for all $R > 0$:

$$\begin{aligned} \int (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}) \rangle d^3x &= \lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \varphi_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x \\ &+ \lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \zeta_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x . \end{aligned}$$

Using Lemma 3.4(c)i), the second integral on the right hand side becomes:

$$\begin{aligned} &| \lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \zeta_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x | \\ &\leq \lim_{R \rightarrow \infty} \frac{M}{R} \left\| \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \psi_{\text{out}}(\mathbf{x}, t) d^3x \right\| \\ &= \lim_{R \rightarrow \infty} \frac{M}{R} \left\| \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \psi_{\text{out}}(\mathbf{x}, 0) d^3x \right\| = 0 . \end{aligned}$$

Therefore:

$$\begin{aligned}
& \int (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), \psi(\mathbf{x}) \rangle d^3x \\
&= \lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \varphi_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x \\
&= \lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty} e^{iEt} \int_{\mathbb{R}^3 \setminus B(0,R)} (2\pi)^{-\frac{3}{2}} \langle \varphi_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x \\
&= \lim_{R \rightarrow \infty} \lim_{t \rightarrow \infty} e^{iEt} \int (2\pi)^{-\frac{3}{2}} \langle \varphi_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, t) \rangle d^3x \\
&= \lim_{R \rightarrow \infty} \int (2\pi)^{-\frac{3}{2}} \langle \varphi_{\mathbf{k}}^s(\mathbf{x}), \psi_{\text{out}}(\mathbf{x}, 0) \rangle d^3x = \widehat{\psi}_{\text{out},s}(\mathbf{k}).
\end{aligned}$$

and (41) is proved.

4.4 Proof of Lemma 2.2

First we want to prove " \Rightarrow ":

Let $\widehat{\psi}_{\text{out}}(\mathbf{k}) \in \mathcal{G}$. According to (40) we have for any $n \in \mathbb{N}_0$:

$$\begin{aligned}
H^n \psi(\mathbf{x}) &= \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} H^n \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) \widehat{\psi}_{\text{out},s}(\mathbf{k}) d^3k \\
&= \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} E_k^n \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) \widehat{\psi}_{\text{out},s}(\mathbf{k}) d^3k.
\end{aligned}$$

Since $\widehat{\psi}_{\text{out}}(\mathbf{k})$ decays faster than any polynomial, this term is bounded and in $L^2 \otimes \mathbb{C}^4$ for all $n \in \mathbb{N}_0$. As the potential $A \in C^\infty$, also

$$(H - A - \beta m)^n \psi(\mathbf{x}) = \nabla^n \psi(\mathbf{x})$$

is bounded and in $L^2 \otimes \mathbb{C}^4$ for all $n \in \mathbb{N}_0$.

Furthermore we have, using (36) in (40):

$$\begin{aligned}
H^n \psi(\mathbf{x}) &= \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) E_k^n \widehat{\psi}_{out,s}(\mathbf{k}) d^3 k \\
&= \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \varphi_{\mathbf{k}}^s(\mathbf{x}) E_k^n \widehat{\psi}_{out,s}(\mathbf{k}) d^3 k \\
&\quad - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \int A(\mathbf{x}') G_k^+(\mathbf{x} - \mathbf{x}') \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}') d^3 x' E_k^n \widehat{\psi}_{out,s}(\mathbf{k}) d^3 k =: I_1 + I_2 .
\end{aligned}$$

I_1 is the Fourier transform of $E_k^n \widehat{\psi}_{out,s}(\mathbf{k})$. As $E_k^n \widehat{\psi}_{out,s}(\mathbf{k}) \in \mathcal{G}$, I_1 lies in \widehat{G} .

Next we write for I_2 :

$$\begin{aligned}
I_2 &= - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \int A(\mathbf{x}') e^{ikx + ik(|\mathbf{x} - \mathbf{x}'| - x)} \frac{S_k^+(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}') d^3 x' E_k^n \widehat{\psi}_{out,s}(\mathbf{k}) d^3 k d\Omega \\
&= - \sum_{s=1}^2 \int \int_0^\infty (2\pi)^{-\frac{3}{2}} \int A(\mathbf{x}') e^{ikx} F(\mathbf{k}, \mathbf{x}, \mathbf{x}') d^3 x' E_k^n \widehat{\psi}_{out,s}(\mathbf{k}) k^2 dk d\Omega
\end{aligned}$$

where

$$F(\mathbf{k}, \mathbf{x}, \mathbf{x}') := e^{ik(|\mathbf{x} - \mathbf{x}'| - x)} \frac{S_k^+(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}') . \quad (106)$$

We make now two partial integrations under the k -integral, which is possible by Fubini's theorem:

$$\begin{aligned}
I_2 &= - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \int_0^\infty \int \mathcal{A}(\mathbf{x}') e^{ikx} F(\mathbf{k}, \mathbf{x}, \mathbf{x}') d^3 x' E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k}) dk d\Omega \\
&= - \sum_{s=1}^2 \frac{1}{x^2} \int (2\pi)^{-\frac{3}{2}} \int_0^\infty \int \mathcal{A}(\mathbf{x}') e^{ikx} \partial_k^2 (F(\mathbf{k}, \mathbf{x}, \mathbf{x}') d^3 x' E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k})) dk d\Omega \\
&= - \sum_{s=1}^2 \frac{1}{x^2} \int \int_0^\infty \int (2\pi)^{-\frac{3}{2}} \mathcal{A}(\mathbf{x}') e^{ikx} \partial_k^2 F(\mathbf{k}, \mathbf{x}, \mathbf{x}') E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k}) \\
&\quad + 2\partial_k F(\mathbf{k}, \mathbf{x}, \mathbf{x}') \partial_k (E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k})) dk d\Omega d^3 x' \\
&\quad - \sum_{s=1}^2 \frac{1}{x^2} \int_0^\infty \int (2\pi)^{-\frac{3}{2}} \int \mathcal{A}(\mathbf{x}') e^{ikx} F(\mathbf{k}, \mathbf{x}, \mathbf{x}') d^3 x' \partial_k^2 (E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k})) dk d\Omega \\
&=: I_3 + I_4 .
\end{aligned}$$

For I_4 we can write, using the definition of F (106) and (36)

$$\begin{aligned}
x^2 I_4 &= \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \widetilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}) \partial_k^2 (E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k})) \frac{1}{k^2} d^3 k \\
&\quad - \sum_{s=1}^2 \int (2\pi)^{-\frac{3}{2}} \varphi_{\mathbf{k}}^s(\mathbf{x}) \partial_k^2 (E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k})) \frac{1}{k^2} d^3 k .
\end{aligned}$$

As $\widehat{\psi}_{out} \in \mathcal{G}$, $\partial_k^2 (E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k})) \frac{1}{k^2}$ lies in L^2 and so does $x^2 \partial_k^n I_4$ for $n \in \mathbb{N}_0$.

Under the \mathbf{k} -integral in I_3 one more partial integration is possible.

$$I_3 = - \sum_{s=1}^2 \frac{1}{x^3} \int (2\pi)^{-\frac{3}{2}} \int \mathcal{A}(\mathbf{x}') \widetilde{F}(\mathbf{k}, \mathbf{x}, \mathbf{x}') d^3 x$$

where

$$\widetilde{F}(\mathbf{k}, \mathbf{x}, \mathbf{x}') := \partial_k (\partial_k^2 F(\mathbf{k}, \mathbf{x}, \mathbf{x}') E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k}) + 2\partial_k F(\mathbf{k}, \mathbf{x}, \mathbf{x}') \partial_k (E_k^n k^2 \widehat{\psi}_{out,s}(\mathbf{k}))) .$$

Due to Lemma 3.4 (c) $\partial_k^n \widetilde{\varphi}_{\mathbf{k}}(\mathbf{x}') \leq Mx'$. Furthermore we have, that

$$| \partial_k e^{ik(|\mathbf{x}-\mathbf{x}'|-x)} | = | (|\mathbf{x}-\mathbf{x}'|-x) e^{ik(|\mathbf{x}-\mathbf{x}'|-x)} | \leq x' | e^{ik(|\mathbf{x}-\mathbf{x}'|-x)} | .$$

It follows, that (remember the definition of F (106))

$$\| \tilde{F}(\mathbf{x}, \mathbf{x}') \|_{s \leq} M_2 \frac{x'^3}{|\mathbf{x} - \mathbf{x}'|}.$$

So due to (48), with Condition B (9) on the potential, the integral

$$\int \mathcal{A}(\mathbf{x}') \tilde{F}(\mathbf{k}, \mathbf{x}, \mathbf{x}') d^3 x'$$

decays as fast as or faster than x^{-1} , so $x^4 I_3$ is bounded. It follows, that $x^2 I_3$ lies in L^2 for $n \in \mathbb{N}_0$.

The proof, that $x \partial_x^n \psi \in L^2$ is similar as above, just with one partial integration less. It follows, that $\psi \in \widehat{\mathcal{G}}$.

It is left to prove "⇐":

By Lemma 3.4(b) it follows, that

$$\begin{aligned} E_k \widehat{\psi}_{out,s}(\mathbf{k}) &= H \widehat{\psi}_{out,s}(\mathbf{k}) = \int (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), H \psi(\mathbf{x}) \rangle d^3 x \\ &= \int (2\pi)^{-\frac{3}{2}} \langle \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), (H_0 + \mathcal{A}) \psi(\mathbf{x}) \rangle d^3 x. \end{aligned}$$

For $\psi \in \widehat{\mathcal{G}}$, the right hand side is integrable, so $E_k \widehat{\psi}_{out,s}(\mathbf{k})$ is bounded. As $\mathcal{A} \in C^\infty$, this can be repeated, so $E_k^n \widehat{\psi}_{out,s}(\mathbf{k})$ is bounded for any $n \in \mathbb{N}$.

Since $E_k = \sqrt{k^2 + m^2} \geq k$, it follows, that

$$k^n \widehat{\psi}_{out,s}(\mathbf{k}) < \infty.$$

Equivalently we get:

$$\begin{aligned} E_k^n \partial_k^j \widehat{\psi}_{out,s}(\mathbf{k}) &= \int (2\pi)^{-\frac{3}{2}} \langle \partial_k^j \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), H^n \psi(\mathbf{x}) \rangle d^3 x \\ E_k^n k^{|\gamma|-1} D_{\mathbf{k}}^\gamma \widehat{\psi}_{out,s}(\mathbf{k}) &= \int (2\pi)^{-\frac{3}{2}} \langle k^{|\gamma|-1} D_{\mathbf{k}}^\gamma \tilde{\varphi}_{\mathbf{k}}^s(\mathbf{x}), H^n \psi(\mathbf{x}) \rangle d^3 x \end{aligned}$$

With (c) of Lemma 3.4 it follows, that for $\psi \in \widehat{\mathcal{G}}$ these terms are bounded for $j = 1, 2$, $n \in \mathbb{N}_0$ and $|\gamma| \leq 2$. So $\widehat{\psi}_{out,s}(\mathbf{k}) \in \mathcal{G}$.

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