

Adiabatic Pair Creation in Heavy Ion and Laser Fields

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The planned generation of lasers and heavy ion colliders renews the hope to see electron-positron pair creation in strong classical fields. This old prediction is usually referred to as spontaneous pair creation. We observe that both—heavy ion collisions and pair creation in strong laser fields—are instances of the theory of adiabatic pair creation. We shall present the theory, thereby correcting earlier results. We give the momentum distribution of created pairs in overcritical fields. We discuss carefully the proposed experimental verifications and conclude that pure laser based experiments are highly questionable. We propose a new experiment, joining laser fields and heavy ions, which is feasible with present day technology and which may indeed verify the theoretical prediction of adiabatic pair creation. Our presentation relies on recent rigorous works in mathematical physics.

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I. INTRODUCTION

The creation of an electron positron pair in an almost stationary very strong external electromagnetic field (a potential well) is often referred to as spontaneous pair creation ([1],[2],[3]). This adiabatic phenomenon emerges straightforwardly from the Dirac sea interpretation of negative energy states: An adiabatically increasing field (a potential well of changing deepness) lifts a particle from the sea to the positive energy subspace (by the adiabatic theorem) where it hopefully scatters and when the potential is switched off one has one free electron and one unoccupied state—a hole—in the sea [4]-[6] (see figure I). A better terminology—and the one we shall use here—is thus adiabatic pair creation (APC).

The figure shows the bound state energy curve of a bound state Φ emerging from the Dirac sea (vacuum), changing with time s due to the change of the potential well. If the potential reaches a high enough value (critical value) the bound state enters the upper continuum and scatters and if it has enough time to escape from the range of the potential, before the potential decreases under its critical value a pair will be created. The experimental realization of APC has been extensively debated for heavy ion collision. We note already here that the theoretical descriptions concerning rates and line shapes were false. The main reason is that no coherent theoretical approach had been found. We give more details later.

Pair creation is also hotly debated in classical laser field technology nowadays [2, 3, 7–9]. But surprisingly the theoretical work on pair creation in classical laser fields

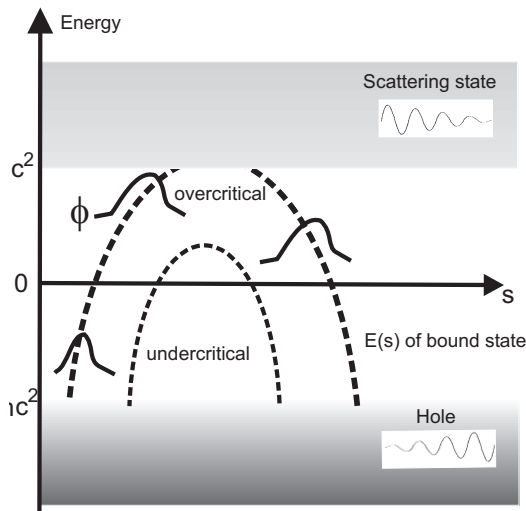


FIG. 1: Schematic presentation of the adiabatic pair creation. It depicts the spectrum of the Dirac operator $D_{\mu(s)}$ as function of s . Depending on the strength of the potential there may exist bound state energy curves $E(s)$, one or more of which may bridge the spectral gap (overcritical case). Also schematically drawn are bound states Φ at various undercritical times.

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which is universally referred to as Spontaneous Pair Creation made no contact with adiabatic theory which in fact it is. Indeed most of the theoretical work is questionable and with that the hope for experimental feasibility. We shall point out where present day arguments are dubious (reviewing the relevant literature) and we shall present an argument and a new, well founded experimental possibility for pair creation in a field which is a combination of an ion field and a classical laser field. We believe that this is in fact the only possible experimental realization feasible in the near future. We shall give the details after our theoretical treatment of APC which we do next.

II. ADIABATIC PAIR CREATION

What we present here is the heuristic core of a mathematical proof of APC (c.f. [10, 11]). This proof while being mathematically and technically very involved is helpful to find confidence in the heuristic short argument we shall give. The mathematical proof changes the quantitative statements only slightly.

Consider the one particle Dirac equation with external electromagnetic field. On microscopic time- and space-scales

$$\tau = \frac{mc^2}{\hbar}t = \frac{c}{\lambda_C}t \quad \mathbf{x} = \frac{mc}{\hbar}\mathbf{r} = \frac{\mathbf{r}}{\lambda_C} \quad (1)$$

the equation reads

$$\begin{aligned} i\frac{\partial\psi_\tau}{\partial\tau} &= -i\sum_{l=1}^3\alpha_l\partial_l\psi_\tau + A_{\varepsilon\tau}(\mathbf{x})\psi_\tau + \beta\psi_\tau \\ &\equiv (D^0 + A_{\varepsilon\tau}(\mathbf{x}))\psi_\tau = D_{\varepsilon\tau}\psi_\tau \end{aligned} \quad (2)$$

where ε is the adiabatic parameter, representing the slow time variation of the external potential and Amc^2 gives the potential in the units eV (we discuss later physical values for ε).

We consider the Dirac equation on the macroscopic time scale $s = \varepsilon\tau$:

$$i\frac{\partial\psi_s}{\partial s} \equiv \frac{1}{\varepsilon}D_s\psi_s. \quad (3)$$

The spectrum of the Dirac operator without external field is $(-\infty, -1] \cap [1, \infty)$. The adiabatic theorem ensures[12] (for ε small) that the gap can only be crossed by bound states Φ_s of the Dirac operator $D_s\Phi_s = E_s\Phi_s$ for which E_s is a curve crossing the gap (see figure). Let us call the curve a gap-bridge. *If there is no gap-bridge the probability of pair creation is exponentially small in $1/\varepsilon$.* It is very important to take note of this: No matter how steep the potential well is, i.e. no matter how strong the "force" is, if there is no gap-bridge pairs will only be created with exponentially small rate.

We assume now, that a gap-bridge exists and consider the bound state Φ_0 (assumed to be non degenerate) at the crossing.

We expand it in generalized eigenfunctions which of course depend also on the "parameter" s . We shall need the eigenfunctions for times σ close to the critical time $\sigma = 0$.

Consider the eigenvalue equation

$$D_\sigma\varphi = E\varphi \quad (4)$$

for fixed $\sigma \in \mathbb{R}$. The continuous subspace is spanned by generalized eigenfunctions $\varphi^j(\mathbf{k}, \sigma, \mathbf{x})$, $j = 1, 2, 3, 4$, with energy $E = \pm E_k = \pm\sqrt{k^2 + 1}$. For ease of notation we will drop the spin index j in what follows.

The generalized eigenfunctions also solve the Lippmann Schwinger equation

$$\begin{aligned} \varphi(\sigma, \mathbf{k}, \mathbf{x}) &= \varphi_0(\mathbf{k}, \mathbf{x}) \\ &+ \int G_k^+(\mathbf{x} - \mathbf{x}')A_\sigma(\mathbf{x}')\varphi(\sigma, \mathbf{k}, \mathbf{x}')d^3x', \end{aligned} \quad (5)$$

with $\varphi_0(\mathbf{k}, \mathbf{x}) = \xi(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$, the generalized eigenfunctions of the free Dirac operator D^0 , i.e. G_k^+ is the kernel of $(E_k - D^0)^{-1} = \lim_{\delta \rightarrow 0}(E_k - D^0 + i\delta)^{-1}$ [13].

Introducing the operator T_σ^k

$$T_\sigma^k f = \int G_k^+(\mathbf{x} - \mathbf{x}')A_\sigma(\mathbf{x}')f(\mathbf{x}')d^3x', \quad (6)$$

(5) becomes

$$(1 - T_\sigma^k)\varphi(\sigma, \mathbf{k}, \cdot) = \varphi_0(\mathbf{k}, \cdot). \quad (7)$$

Note that

$$(1 - T_0^0)\Phi_0 = 0. \quad (8)$$

We estimate the propagation of a wave function generated by the *static* Dirac Operator $D_\sigma = D^0 + A_\sigma(\mathbf{x})$, where $\sigma > 0$ should be thought of as near the critical value (the relevant regime turns out to be of order $\sigma = \mathcal{O}(\varepsilon^{1/3})$).

Since the generalized eigenfunctions for $(\sigma, \mathbf{k}) \approx (0, 0)$ are close to the bound state Φ_0 it is reasonable to write in leading order (see section II A for more explanations):

$$\varphi(\sigma, \mathbf{k}, \mathbf{x}) \approx \eta_\sigma(\mathbf{k})\Phi_0(\mathbf{x}). \quad (9)$$

Since they solve (5), the first summand of (5) must become negligible with respect to $\eta_\sigma(\mathbf{k})\Phi_0$, which is part of the second summand. Hence $\eta_\sigma(\mathbf{k})$ must diverge for $(\sigma, \mathbf{k}) \rightarrow (0, 0)$. For the outgoing asymptote of the state Φ_0 (generalized Fourier transform) evolved with D_σ near criticality we have with (9) that

$$\begin{aligned} \widehat{\Phi}_{out}(\sigma, \mathbf{k}) &:= \int (2\pi)^{-\frac{3}{2}}\Phi_0(\mathbf{x})\overline{\varphi}(\sigma, \mathbf{k}, \mathbf{x})d^3x \\ &\approx (2\pi)^{-\frac{3}{2}}\overline{\eta}_\sigma(\mathbf{k}). \end{aligned} \quad (10)$$

Now, for (σ, k) close to but different from $(0, 0)$, $\eta_\sigma(\mathbf{k}) \sim \widehat{\Phi}_{out}(\sigma, \mathbf{k})$ will be peaked around a value $k(\sigma)$ with width $\Delta(\sigma)$ (determined below) defined by

$$\eta_\sigma(k(\sigma) \pm \Delta(\sigma)) \approx \eta_\sigma(k(\sigma))/\sqrt{2}. \quad (11)$$

We may use the width for the rough estimate

$$|\partial_k \widehat{\Phi}_{out}(\sigma, \mathbf{k})| < \Delta(\sigma)^{-1} \widehat{\Phi}_{out}(\sigma, k_\sigma), \quad (12)$$

where the right hand side should be multiplied by some appropriate constant which we — since it is not substantial — take to be one. Using (9), (10), $d^3k = k^2 d\Omega dk$ and partial integration (observing $\frac{\varepsilon}{-iks} \partial_k e^{-i(1+\frac{k^2}{2})\frac{s}{\varepsilon}} = e^{-i(1+\frac{k^2}{2})\frac{s}{\varepsilon}}$) we get

$$\begin{aligned} U_\sigma(s, 0)\Phi_0 &= e^{-isD_\sigma}\Phi_0 \approx \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{-i(1+\frac{k^2}{2})\frac{s}{\varepsilon}} \widehat{\Phi}_{out}\varphi d^3k \\ &= \frac{-i\varepsilon}{s} \int e^{-i(1+\frac{k^2}{2})\frac{s}{\varepsilon}} \partial_k \left(|\widehat{\Phi}_{out}|^2 \Phi_0(x) k d\Omega \right) dk \end{aligned}$$

By (12), (10) and (11), assuming that $\Delta(\sigma) \ll k(\sigma)$

$$\begin{aligned} |\partial_k \left(|\widehat{\Phi}_{out}|^2 k \right)| d\Omega dk &\approx |\widehat{\Phi}_{out}|^2 \left(\frac{2}{\Delta(\sigma)k} + \frac{1}{k^2} \right) d^3k \\ &\approx |\widehat{\Phi}_{out}|^2 \frac{2}{\Delta(\sigma)k(\sigma)} d^3k. \end{aligned}$$

Hence

$$|U_\sigma(s, 0)\Phi_0(\mathbf{x})| \leq \frac{2\varepsilon|\Phi_0(x)|}{s\Delta(\sigma)k(\sigma)} \int |\widehat{\Phi}_{out}|^2 d^3k.$$

Since Φ_0 is normalized we get for the decay time s_d , defined by $|\langle U(s_d, 0)\Phi_0, \Phi_0 \rangle| \approx 1/2$

$$s_d \approx 4\varepsilon(k(\sigma)\Delta(\sigma))^{-1}. \quad (13)$$

The important information we must provide is thus $\eta_\sigma(\mathbf{k})$ for $\sigma \approx 0$. In view of (7) and (9) we have that

$$A_\sigma(1 - T_\sigma^k)\eta_\sigma(\mathbf{k})\Phi_0 \approx A_\sigma\varphi_0(\mathbf{k}, \cdot). \quad (14)$$

We can estimate $\eta_\sigma(\mathbf{k})$ by considering the scalar product of (14) with Φ_0 :

$$\eta_\sigma(\mathbf{k})\langle (1 - T_\sigma^k)\Phi_0, A_\sigma\Phi_0 \rangle \approx \langle \varphi_0(\mathbf{k}, \cdot), A_\sigma\Phi_0 \rangle.$$

One finds that $\langle \varphi_0(\mathbf{k}, \cdot), A_\sigma\Phi_0 \rangle = Ck + \mathcal{O}(k^2)$ with an appropriate $C \neq 0$. Thus

$$\eta_\sigma(\mathbf{k}) \approx Ck \langle (1 - T_\sigma^k)\Phi_0, A_\sigma\Phi_0 \rangle^{-1}.$$

Expanding T_σ^k in orders of k around $k = 0$ until fourth order yields (the first order term turns out to be zero on general grounds ([10],[11]))

$$\eta_\sigma(\mathbf{k}) \approx \frac{-Ck}{C_0\sigma - (|C_2| + \mathcal{O}(\sigma))k^2 - i(|C_3| + \mathcal{O}(\sigma))k^3} \quad (15)$$

For $C_0\sigma \approx C_2k^2$ the denominator behaves like C_3k^3 , otherwise it behaves like $C_0\sigma - C_2k^2$. Hence by (10)

$$|\widehat{\Phi}_{out}(\sigma, \mathbf{k})|^2 \approx Ck^2 \left((C_0\sigma - |C_2|k^2)^2 + |C_3|^2k^6 \right)^{-1} \quad (16)$$

This result [10] differs from the results given in the literature (see e.g. formula (6.37) in [1] and formula (7) in [14], where it should be noted that we discuss the momentum distribution and not the energy distribution as is done in [1, 14]). The right hand side of (16) obviously diverges for $(\sigma, k) \rightarrow (0, 0)$. For fixed $0 \neq \sigma \approx 0$ the divergent behavior is strongest close to the resonance at $(C_0\sigma - |C_2|k(\sigma))^2 = 0$

$$k(\sigma) = \sqrt{\sigma C_0|C_2|^{-1}}. \quad (17)$$

In view of (11) $\Delta(\sigma)$ can be roughly estimated by setting the right hand side of (16) equal to 1/2 of its maximal size, i.e.

$$C_0\sigma - |C_2|(k(\sigma) + \Delta(\sigma))^2 \approx |C_3|k^3(\sigma)$$

hence

$$\Delta(\sigma) \approx k(\sigma)^2 |C_3|(2|C_2|)^{-1}. \quad (18)$$

For a rough estimate of the decay time s_d we set $\sigma = s_d$ and use (13), (17) and (18). This yields in units of τ (cf. (1))

$$s_d^{\frac{5}{2}} = \frac{8\varepsilon|C_2|^{\frac{5}{2}}}{|C_0|^{\frac{3}{2}}|C_3|}. \quad (19)$$

A. Long Range Potentials

Our prediction (15) is based on so called short range potentials, i.e. potentials which fall off faster than x^{-2} . For long range potentials (e.g. Coulomb) Popov showed [15] that the width (and thus the decay time) of the spontaneously created positron is exponentially small $\sim e^{-1/k(\sigma)}$. It may be interesting to see how Popov's result relates to our procedure. For that one must consider the error in replacing $\varphi(\sigma, k, x)$ by $\eta(\sigma, k)\Phi_0$ (c.f. (9)), i.e.

$$\varphi(\sigma, k, x) := \eta(\sigma, k)\Phi_0 + \zeta(\sigma, k, x).$$

It is reasonable to assume that $\zeta(\sigma, k, 1) = \mathcal{O}(1)$, but with (15) $\eta(\sigma, k(\sigma)) = \mathcal{O}(1/k(\sigma)^2)$ hence $\zeta(\sigma, k, x) = \mathcal{O}(k^2\eta(\sigma, k))$ which is actually true for short range potentials [10].

For the case of long range potentials one finds from the analysis in [10] that for Coulomb $\zeta(\sigma, k, x) \sim \mathcal{O}(k\eta(\sigma, k))$. The argument is not difficult but relies a bit on the theory of generalized eigenfunctions and uses the self adjointness of $A_\sigma T_\sigma^k$. Following now the procedure under (9) we obtain the expansion

$$\begin{aligned} (1 - T_\sigma^k)\varphi(\sigma, k, x) &= \eta(\sigma, k)\Phi_0(x)(C_0\sigma + C_2k^2 + C_3k^3) \\ &\quad + (\tilde{C}_0\sigma + \tilde{C}_2k^2)\zeta(\sigma, k), \end{aligned}$$

where $\zeta(\sigma, k)$ should be thought of as the order of magnitude of $\zeta(\sigma, k, x)$. Observing that in the Coulomb case

$\zeta(\sigma, k, x) = \mathcal{O}(k\eta(\sigma, k))$ we see that the η -width-term $iC_3k(\sigma)^3$ can be canceled by the ζ -contribution and thus the width of the resonance can in principle be arbitrarily small. We give now a separate argument for how small. Let the radius of the attractive inner part of the potential be r . Then by the zero penetrability of the Coulomb-barrier $\Phi(r) \approx 0$, thus

$$\varphi(\sigma, k, r) \approx \zeta(\sigma, k, r) = \mathcal{O}(k\eta(\sigma, k))$$

while

$$\varphi'(\sigma, k, r) \approx -|\Phi'(r)|\eta(\sigma, k)$$

hence

$$\varphi'(\sigma, k, r)/\varphi(\sigma, k, r) = -|\mathcal{O}(1/k)|,$$

thus $\zeta(\sigma, k, r)$ is of order $e^{-1/k}$, which is in line with the result of Popov [15].

III. EXPERIMENTS

A. Heavy Ion Collisions

We turn now to the experimental verifications and prior discussions in the theoretical physics literature. The experimental verification of APC has been sought in heavy ion collisions (HIC) (but without success so far [16], [17]). Here the adiabatic time scale on which the field increases is directly determined by the relative speed with which the heavy ions approach each other and one computes that ε is of order 10^{-1} [1]. Theoretical work on HIC was extensively done [1, 14, 15, 18, 19]. The realistic case of Coulomb scattering has been analyzed by Popov showing that adiabatic pair creation for slight overcriticality is exponentially small. This is the situation which is believed to be at best achievable in HIC [1].

B. Laser-Pair-Creation

Another experimental situation with adiabatically changing fields is provided by lasers. For laser fields (wavelength λ) $\varepsilon = \lambda_C/\lambda \approx 10^{-6}$, where λ_C is the Compton wave length of the electron. It is in fact hoped, that pair creation can be seen in a new generation of lasers which are able to create in focus a very well localized overcritical classical field. Unfortunately this hope is intermingled with misconceptions which we shall try to sort out. Let us review shortly the present status of the discussion on the possibility of creating pairs in laser fields. An early computation by Schwinger [20] predicted a pair creation rate for a constant strong electric field. In that case the rate for small fields is exponentially small. This is because pair creation arises in a similar manner as in the Klein paradox tunnelling, where “negative energy states” overlap with their exponentially small tails the

“positive energy states” [21]. One can picture the situation by “tilting” the spectral gap (of figure I) under the influence of the electric field) so that wave functions “can reach across”. As a rule of thumb the critical electric field E_c which one needs is estimated by the potential energy the electron acquires over a distance of roughly the Compton-wave length: $eE_c\lambda_C = 2mc^2$. The Schwinger computation is certainly correct but of course has nothing to do with lasers. The Schwinger-computation has been generalized by Brezin and Itzykson to time varying but spatially constant electric fields [22]. That generalization did not change the rates for pair creation as computed by Schwinger. They are still determined by Klein-paradox tunnelling across the “tilted” spectral gap. But Brezin and Itzykson apply now their results to laser fields. Their argument is, that while laser fields are not of the type of the electric potential field they treat (a field of a huge capacitor with plates infinitely far apart) since they have magnetic parts as well, their field strength should be at least a lower bound to the strength of a realistic laser field which would be able to create pairs. That the magnetic part of the laser field can be neglected is not much argued for, except that it is bluntly stated that pure magnetic fields cannot create pairs. In [23] one finds a perturbation argument from which it is also concluded that pair creation is an “electric effect”. As a note aside it is also folklore wisdom that a single electro magnetic wave cannot produce pairs, it is therefore standardly assumed that at least two laser fields are superposed so that in the focal region a standing wave is formed. The Brezin-Itzykson reduction of a laser field to an electric potential field became the basis of all further theoretical research on laser pair creation (see e.g [2, 3, 7]).

What is wrong with all that? Firstly we recall that the Brezin-Itzykson treatment was intended to discourage hopes for seeing pair creation with optical lasers. They actually say that they would not have trusted their bound if it would have turned out lower than what they got. The moral which should be drawn is simply: The true lower bound can and will be much higher than what Brezin-Itzykson found. In fact we shall argue later that no lower bound exists. In reverse, it is not a priori clear at all that pair creation is determined by the maximal value of the present \vec{E} -field. What is clear however is that that treatment gives wrong results when applied to a single overcritical laser beam, where no pair creation occurs. Therefore if the Brezin-Itzykson treatment is thought to be of relevance for superposed laser fields an argument is truly needed, but none has been given.

Secondly we wish to emphasize the obvious, namely that the talk about electric fields in the context of quantum mechanics and in particular in the context of pair creation can be misleading: Only the four-vector-potential enters in the Dirac equation. Descriptions in terms of the *local* behavior of the E - or B - fields may therefore lead off the track. (This is similar to the role of the A and B field in the Aharonov Bohm effect). In particular laser fields must not be modelled by a constant

electric field. The A -fields with $A_0 = 0$ (in Coulomb gauge) differs from the A -field where only $A_0 \neq 0$ (in Coulomb gauge) in all respects. In particular it is important to take the full geometry of the laser field into account. One moral of APC is that pair creation is a “global” effect, i.e. the global \mathbf{x} and t dependence of the field plays a role: Criticality has to be defined via the bound states of the *potential*. Whether there is a bound state at the edge or not depends on the full \mathbf{x} -dependence on that field, not on the maximal \vec{E} -field only. In APC the pair creation behavior of an (slightly) overcritical external potential is qualitatively different from the pair creation behavior of an (slightly) undercritical potential, though their “local” \vec{E} - and \vec{B} -fields (in particular their maximal \vec{E} -field) may be almost the same. That clearly contradicts the assumption that the pair creation rate is given by the maximal \vec{E} -field only. A related point arising from the global nature of APC is that perturbative treatments are highly questionable [24].

Let us now return to the crucial question: Does there exist a gap-bridge for laser fields? The electric part of a laser field is not a gradient of an electric potential and unfit for the job to lift a ground state from the lower to the upper continuum. What about taking the laser field seriously, considering also the magnetic part of the field? The magnetic field, in particular the *spin-magnetic field interaction* provides in fact an adiabatically changing potential as APC requires. Unfortunately that does not work either. For constant \vec{B} -fields it is already known that no gap-bridge exists (see [25] and [26] for a rigorous statement). We give the simple argument now for the more general case of fields with varying magnitude but fixed direction. The argument shows that typically a gap-bridge does not exist for the Dirac equation *without electric potential* ($A_0 = 0$ in Coulomb gauge), i.e. that they do not exist for laser fields. In fact we argue that there is no bridge crossing the spectral value zero! Starting with the eigenvalue equation for the Dirac operator and writing $\underline{A} = \vec{A} \cdot \vec{\alpha} = \sum_{j=1}^3 A_j \alpha_j$ we compute

$$\begin{aligned} (D_0 + \underline{A})\Phi &= (E - A_0)\Phi \\ (D_0 + \underline{A})^2\Phi &= (D_0 + \underline{A})(E - A_0)\Phi \\ ((i\nabla - \underline{A})^2 + 1)\Phi &= (i(\nabla A_0) + (E - A_0)^2)\Phi. \end{aligned}$$

This then yields

$$(H + m^2 + i\nabla \underline{A})\Phi = (i(\nabla A_0) + (E - A_0)^2)\Phi$$

where H is the Schrödinger operator with vector potential $\vec{A} = (A_1, A_2, A_3)$ in Coulomb gauge ($\text{div} \vec{A} = 0$). By simple α -algebra we have that

$$\nabla \underline{A} = \sum_{j,k} \alpha_j \alpha_k \partial_j A_k = \text{div} \vec{A} \mathbb{E}_4 + \underline{B} \alpha_4 = \underline{B} \alpha_4$$

where \mathbb{E}_4 is the 4×4 -unit matrix,

$$\alpha_4 = \alpha_1 \alpha_2 \alpha_3 = \begin{pmatrix} 0 & \mathbb{E}_2 \\ \mathbb{E}_2 & 0 \end{pmatrix}$$

and $\underline{B} = \vec{B} \cdot \vec{\alpha}$. Now we assume that the magnetic field \vec{B} has constant direction (as it is the case in a standing wave), i.e. $\underline{B}(\vec{x})\alpha_4 = b(\vec{x})\underline{B}_0\alpha_4$ with constant matrix \underline{B}_0 and real function $b(\vec{x})$ giving the strength of the magnetic field.

Using this and $A_0 = 0$ we have that

$$(H + 1 + b(\vec{x})\underline{B}_0\alpha_4)\Phi = E^2\Phi$$

We go to the eigenspaces of the matrix $\underline{B}_0\alpha_4$ setting $\varphi_j = \langle s_j, \Phi \rangle$. Note here, that α_4 commutes with α_l for $l = 1, 2, 3$ and that the eigenvalues of the matrix $\underline{B}_0\alpha_4$ are $\mu = \pm 1$

$$(H + 1 + \mu b(\vec{x}))\varphi_j = E^2\varphi_j. \quad (20)$$

(Note aside that in physical units μ is the magnetic moment.) From this we see that the energy curve of the Schrödinger operator $H + 1 + \mu b(\vec{x})$ must be quadratically zero if there is a gap bridge of the corresponding Dirac operator crossing zero. This however is a very special behavior of eigenvalues and thus unlikely. (See [25] where a famous special case is actually discussed.) One may conclude that gap bridges do not occur at all for laser fields. On the other hand if $A_0 \neq 0$ this argument breaks down and a gap bridge may exist.

Our first conclusion is thus: No matter how tricky laser waves are superposed and how strong the fields may be, no gap-bridge arises and pair creation will at most be exponentially small in $1/\varepsilon$. Note however, that we made an external field approximation in our treatment. Considering self interaction effects, in particular the anomalous magnetic moment of the electron, a gap bridge may arise for \vec{B} fields which are more than a few thousand times overcritical [27].

Our second conclusion is however: Since the magnetic field in junction with a Coulomb field can produce a gap bridge, a new (and possibly the only one) experimental possibility for APC arises. We propose to combine laser- and heavy ion fields, i.e. to shoot heavy ions into the focus of the laser. We model the laser field by an oscillating magnetic field \vec{B} constant in space. First let us explain, why this model is satisfied for the present situation. As mentioned above, descriptions in terms of \vec{E} and \vec{B} fields may be misleading, so let us model the vector potential of the laser. In Coulomb-gauge the vector potential has no A_0 -component. Since the vector potential of an optical laser varies for a given time only slightly over the range of the heavy ion potential the laser-field is in good approximation given by its first order Taylor expansion around the center of the nucleus. Note, that in Coulomb-gauge the zero-order term of the vector potential is zero. Hence the leading order term of the vector potential is the first order term, which has (in Coulomb gauge) no gradient and can thus be written in the form $\vec{B} \times \vec{x}$ for some \vec{B} constant in space, oscillating in time. Thus the vector potential of the laser is in good approximation the potential of a spatially constant magnetic field in the situation at hand.

Ion	Z	lower bound	numerical result
Ca	20	≈ 40	$\approx 10^9$
Zr	40	≈ 10	$\approx 10^4$
U	92	≈ 2	≈ 100

TABLE I: The table gives an estimated lower bound [26] and a numerical result [28] of the magnetic field strength (in units of E_c/c) for a gap bridge when a nucleus with charge Z is present.

In [26] an estimate (actually a lower and upper bound) for the the critical strength of a (spatially constant) magnetic field which has a gap bridge in presence of a nucleus with charge Z is given. Instead of estimating the lower bound on the \vec{B} -field needed in junction with the ion field to have APC we translate the corresponding lower bounds to the laser electric field strength E . This enables us to have easy comparison with the results given by the laser community. We should however warn the reader that we use here the the lower bound of [26] because it is computable number, while the (reliable) upper bound is not sharply defined. There is also an early numerical result for a bridge gap in a constant magnetic field [28] which suggests that the lower bound is much too low.

In table I we list the values for B for three different nuclei in units $B_c = E_c/c$. We recall that $E_c = \frac{m^2 c^3}{e\hbar}$ (the

critical field strength given by Schwinger) is the critical field strength where the respective B -field in a light wave satisfies $\mu B = mc^2$. For uranium this yields that the field has to be one hundred times overcritical (see table I), if one trusts the numerical result. The enormous difference between rigorous lower bound and the numerics shows that a good rigorous upper bound is most desirable. In this respect we remark that laser fields which are a few times overcritical are expected to be reached by a new generation of lasers, called XFEL (see e.g. [7]). Bringing a heavy ion (like uranium) into the focus of such a laser APC might be observable if strong enough fields can be generated. In fact, the transition from no pairs to periodic (twice per laser period) appearance of pairs when the laser field becomes strong enough is very much like a phase transition. Again the question of predicting the right line shape is interesting and it is reasonable to assume (since the leading order potential is the μB field) that our line shape for short range potentials is correct.

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