

# On Adiabatic Pair Creation

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## Abstract

We give here a rigorous proof of the well known prediction of pair creation as it arises from the Dirac equation with an external time dependent potential. Pair creation happens with probability one if the potential changes adiabatically in time and becomes overcritical, which means that an eigenvalue curve (as function of time) bridges the gap between the negative and positive spectral continuum. The potential can be thought of being zero at large negative and large positive times. The rigorous treatment of this effect has been lacking since the pioneering work of Beck, Steinwedel and Süßmann [2] in 1963 and Gershtein and Zeldovich [8] in 1970.

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# 1 Introduction

Adiabatic pair creation (APC) has been called—unfortunately misleading—spontaneous pair creation (see e.g. [12] and references therein). The creation of electron positron pairs in very strong external classical electromagnetic fields arises straight forwardly from the Dirac sea interpretation of negative energy states. After Dirac[5] it has been discussed as an academic problem by Klein [11], Sauter [34], Heisenberg and Euler [10], Schwinger [35] and Brezin and Itzykson [3]. A more realistic setting was considered by Beck, Steinwedel and Süßmann [2] and worked out by Gershtein and Zeldovich [8] as APC. In the common physics language using the so called Dirac sea picture APC may be shortly described as follows (we explain this below in much greater detail): An adiabatically increasing electric potential lifts a particle from the sea to the positive energy subspace where it scatters and when the potential is gently switched off one has one free electron and one unoccupied state—a hole—in the sea. The experimental verification needs very strong classical fields [12] and is discussed elsewhere [23]. In this respect we would like to remark that a coherent analysis of the existence of APC has been lacking until recently [21]. Earlier quantitative results based on an ad hoc and incoherent analysis (see for example [17, 38, 16, 27, 30, 37]) give an incorrect rate and distribution of outgoing momenta of the spontaneously created pairs (see [23] for a short discussion). For the case of a potential well and for the long range Coulomb potential correct rates have been obtained in [24, 25]. There are also results and discussions in the mathematical physics literature related to APC, notably [18, 19, 26, 36]. But those results are for models which avoid the adiabatic transition of the potential from undercritical to overcritical values and thus do not come to grips with the heart of the problem of APC, which is the control of the wavefunction evolution within the neighborhood of the spectral edge  $mc^2$ .

In APC one considers the so called external field problem, where interactions between the charges are neglected. Vacuum polarization will in general perturb the external field and - using mean field approximation -(see [9]) one may think of the external field as an effective field.

The existence of APC in second quantized external field Dirac theory (if the latter exists<sup>1</sup>) is equivalent to the existence of certain types of solutions of

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<sup>1</sup>It is well known that the lifting of the Dirac evolution (with a smooth field of compact support) to Fock space (second quantisation) is possible if and only if the Shale-Stinespring

the Dirac equation (see e.g. [18] and [21]) which we describe below. The existence of APC in terms of the second quantised S-matrix theory of the Dirac equation with external field is “by definition” equivalent to the existence of these types of solutions of the Dirac equation. We shall in fact formulate our result in terms of the solutions of the Dirac equation and use the Dirac sea picture for the interpretation of the particular solution the existence of which we prove in this paper.

Consider the Dirac equation with an external electric field. Then the potential  $A$  can be chosen as a real valued multiple of the  $4 \times 4$  unit matrix. (We wish to note that the results can be extended to general four potentials. Concerning strong magnetic fields we wish to call attention to the recent work of Dolbeault et.al. [6] as well as [23]).  $Amc^2$  gives the potential in the units eV. We assume that the potential  $A$  varies slowly with time, expressed by  $A_{\varepsilon\tau}$ , where  $\varepsilon$  is a dimensionless small parameter (given by the physics, see [23] for some examples) which in this work will eventually be sent to zero to obtain limit results. Here  $\tau = \frac{mc^2}{\hbar}t$  and  $\mathbf{x} = \frac{mc}{\hbar}\mathbf{r}$  are the dimensionless microscopic time- and space-scales and the Dirac equation in the standard representation reads with the notation  $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$  and  $\partial_l := \frac{\partial}{\partial x_l}$

$$\begin{aligned} i\frac{\partial\psi_\tau(\mathbf{x})}{\partial\tau} &= -i\sum_{l=1}^3\alpha_l\partial_l\psi_\tau(\mathbf{x}) + A_{\varepsilon\tau}(\mathbf{x})\psi_\tau(\mathbf{x}) + \beta\psi_\tau(\mathbf{x}) \\ &=: (D_0 + A_{\varepsilon\tau}(\mathbf{x}))\psi_\tau(\mathbf{x}). \end{aligned} \tag{1}$$

We introduce in (1) the macroscopic time scale  $s = \varepsilon\tau$ . We wish to restrict ourselves to potentials  $A_s$  which can be factorized into a space- and a time dependent factor  $A_s(\mathbf{x}) := A(\mathbf{x})\mu(s)$ , a restriction of technical nature which eases notations and computations and which furthermore helps to picture a spatial potential well, which changes its depth with time. With the new time scale we have

$$i\frac{\partial\psi_s(\mathbf{x})}{\partial s} = \frac{1}{\varepsilon}(D_0 + A(\mathbf{x})\mu(s))\psi_s(\mathbf{x}) =: \frac{1}{\varepsilon}D_{\mu(s)}\psi_s(\mathbf{x}). \tag{2}$$

Furthermore we wish to restrict ourselves to potentials  $A_s(\mathbf{x})$  which are smooth, bounded, compactly supported in  $\mathbf{x}$  and  $s$ , and positive. The spectrum of the free Dirac operator  $D_0$  is absolutely continuous and given by

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condition is satisfied [41], which is the case if and only if the magnetic field vanishes. On the other hand, the S-matrix can always be lifted.

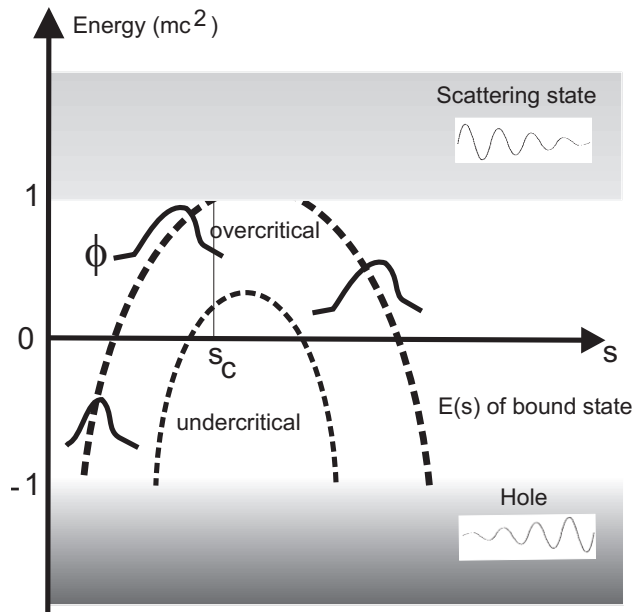


Figure 1: Schematic presentation of the adiabatic pair creation. It depicts the spectrum of the Dirac operator  $D_{\mu(s)}$  as function of  $s$ . Depending on the strength of the potential there may exist bound state energy curves  $E(s)$ , one or more of which may bridge the spectral gap (overcritical case). Also schematically drawn are bound states  $\Phi$  at various undercritical times. No bound states exist in the lower and upper spectral continua  $(-\infty, -1)$  and  $(1, \infty)$ . For more information see the text.

$(-\infty, -1] \cap [1, \infty)$ , defining “negative and positive energy” subspaces. In the Dirac sea interpretation wavefunctions which lie in the positive energy subspace of the free Dirac operator are interpreted as wavefunctions of electrons. The “vacuum” of second quantized Dirac equation corresponds in the Dirac sea picture to all “states of the negative energy subspace being occupied by particles”—the Dirac sea. “Holes” in the Dirac sea are unoccupied negative energy states which are interpreted as anti-electrons, i.e. positrons. The goal of our paper is to assert that there exist solutions of (2) which describe pair creation. We explain what that means looking at Fig. 1.

Consider first the spectrum of the time dependent Dirac operator  $D_{\mu(s)}$ . At large negative and large positive times when  $A_s = 0$ ,  $D_{\mu(s)} = D_0$  and the spectrum is that of  $D_0$ . At times at which  $A_s \neq 0$  there may be eigenvalues in the gap  $[-1, 1]$ , while the continuous spectrum remains unchanged.

The eigenvalues change with the strength of the potential, i.e. with time  $s$  (bound state energy curve  $E(s)$  in Fig. 1). Suppose first that no eigenvalue reaches 1, i.e. no bound state energy curve bridges the gap (undercritical case). The adiabatic theorem (see e.g. [39]) ensures that there is no tunneling across spectral gaps meaning that the bound states stay more or less intact when the potentials changes adiabatically. In terms of solutions of the Dirac equation (2) this means the following: When  $\varepsilon$  goes to zero, there exists no solution which for large negative times lies completely within the negative continuous energy subspace and for large positive times has parts in the positive continuous energy subspace. That means in the Dirac sea interpretation that the probability of creating a pair is zero.

However, when the external field becomes overcritical (at time  $s_c$  in Fig.1), i.e. when the largest eigenvalue curve reaches the positive continuum, the bound state ceases to exist and becomes a continuum state (a “resonance”) in the positive continuum subspace. Then there exists a solution of the Dirac equation which follows adiabatically the path of this bound state, which for large negative times must develop into a wavefunction which lies entirely in the negative continuum energy subspace and for positive times may have a part in the positive continuum energy subspace. As indicated in Fig. 1 when the potential decreases with increasing time there is again a bound state energy curve bridging the gap. In principle the solution of the Dirac equation could have a part which “follows” the bound state back into the negative continuous energy subspace and remains there when the potential is switched off. In the Dirac sea interpretation such a solution of the Dirac equation would correspond to pair creation with a probability determined by the absolute squares of the two parts of the wave function. We show however, that no such “back sliding” is adiabatically possible, i.e. no such solution of the Dirac equation exists. The former bound state scatters in the positive continuum energy subspace, i.e. it stays there for all later times (“Scattering state” in Fig. 1). In the Dirac sea picture the “returning” bound state remains for sure empty and upon becoming a state within the negative continuum energy subspace there is now an unoccupied state in the sea (“Hole” in Fig. 1): APC is accomplished with probability one. One pictorial way to describe APC is to imagine the potential acting as an elevator, lifting a particle from the sea to the “upper”(positive) continuum. The scenario is symmetric under change of sign of the potential: It then transports an unoccupied state (a hole) from the positive continuum to the sea and catches a particle from the sea when it is switched off. The hole (positron) then

scatters.

We understand now the type of solution of (2) we wish to study, namely a solution which at some moment of time at which an overcritical bound state exists is equal to that bound state. The scenario described above translates mathematically into the task to establish scattering for such solutions of the time-inhomogeneous Dirac equation (2). To show to what extent the scenario of APC holds one must control first that the bound states stay on the adiabatic time scale intact until the time  $s_c$  the eigenvalues reach the positive continuum. That is content of an adiabatic lemma without a gap and “relatively easy” to establish. The solution of (2) is thus adiabatically essentially represented by the “time dependent bound states” until  $s_c$ . Then we must control the propagation of the wavefunction (the resonance) emerging from the bound state during over-criticality. We wish to show that it scatters. This task is difficult, since the Dirac operator changes with time. The time evolution will be controlled by generalized eigenfunctions, i.e. by a stationary phase argument, which is not standard because the generalized eigenfunctions themselves depend now on time. But we must also take into account the bad (resonant) behavior of the generalized eigenfunctions near criticality. (We wish to note that also [33] is concerned with the wavefunction propagation for time dependent Hamiltonians but under generic smallness assumptions on the potential, assumptions which are not fulfilled in our problem). They become unbounded for critical  $k$ -values (which are small) and hence the situation is very much different from the usual scattering situation governed by “plane waves” (see [23] for a heuristic argument giving some intuition). As we shall find out, the decay time of a wave, say from a bounded spatial region (i.e. the time in which roughly half of the mass left the region), is now of the order of  $\varepsilon^{-2/3}$  in units of the microscopic time as compared to  $\mathcal{O}(1)$  in the common plane wave scattering situation. This means that the resonance lingers around the range of the potential for a much longer time than in the usual scattering of wavefunctions. Such a metastable state decay has already been suggested by [2].

We shall give in the next section the result: Theorem 2.4 and Corollary 2.5. The rest of the paper is devoted to the proof of the theorem. The proof is technically very involved. Instead of describing here what is in the sections to follow we first give the result and then give in Section 3 a skeleton of the proof with a description of the contents of the sections.

## 2 The Result

We begin with

**Notation 2.1** For any  $\mathbf{a} \in \mathbb{R}^3$  we write its norm as  $|\mathbf{a}| = a$ .

The functions we mainly consider are spinors in the space  $L^n(\mathbb{R}^3, \mathbb{C}^4)$ ,  $n = 1, 2, \infty$ . We shall denote this space if no ambiguity arises simply by  $L^n$ . We shall have two scalar products: (i) For  $a, b \in \mathbb{C}^4$ :  $\bar{a}b := \sum_{\alpha=1}^4 a_\alpha^* b_\alpha$  where  $*$  denotes complex conjugation. For  $a \in \mathbb{C}^4$ :  $|a| = \sqrt{\bar{a}a}$  (ii) For  $\psi, \chi \in L^2$ :  $\langle \psi, \chi \rangle := \int \bar{\psi}(\mathbf{x})\chi(\mathbf{x})d^3x$ ,  $\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$ . We shall also consider the operator norm  $\|\cdot\|_{op}$  of operators  $D$  from  $L^2$  to  $L^2$  given by  $\|D\|_{op} = \sup_{\|\Phi\|=1} \|D\Phi\|$ .

*Warning:* Constants appearing in estimates will generically be denoted by  $C$ . We shall not distinguish constants appearing in a sequence of estimates, i.e. in  $X \leq CY \leq CZ$  the constants may differ.

In the following we will only consider potentials which are bounded, compactly supported, positive and purely electrostatic. The latter implies that  $A$  will be a multiple of the unit matrix (since we stick to one inertial frame throughout the paper). Thus  $A$  can be written as a scalar function. To have the possibility of pair creation the external (scalar) field  $\mu(s)A$  has to become critical for some time  $s$  and the first such time will be set  $s = 0$  and we choose  $\mu(0) = 1$ , i.e.  $A$  is critical. Criticality means for us that the (generalized) eigenvalue equation for energy-eigenvalue 1

$$(D_0 + A)\Phi = \Phi \tag{3}$$

has only bound states solutions (i.e.  $L^2$ -solutions). In general there could be also resonances [15], but generically this is not the case for the critical Dirac operator (in our setting). We shall now collect the conditions in a form most convenient for our considerations.

**Condition 2.2** For  $A : \mathbb{R}^3 \rightarrow \mathbb{R}^+$  and  $\mu : \mathbb{R} \rightarrow \mathbb{R}$  we shall require that

- (i)  $A$  has compact support  $\mathcal{S}_A$ ;  $A, \nabla A$  are bounded and  $A$  is critical. Furthermore  $D_1 = D_0 + A$  has no resonances for  $E = 1$  and  $E = 1$  is an  $n$ -fold degenerate eigenvalue for some  $n \in \mathbb{N}$  with eigenspace denoted by  $\mathcal{N}$ :

$$\mathcal{N} := \{\Phi \in L^2 : (D_0 + A - 1)\Phi = 0\} . \tag{4}$$



- (ii) There exists a  $\delta > 0$  such that for any  $\mu \in [0, 1 + \delta]$  there exist not more than one eigenvalue  $E_\mu$  of the operator  $D_\mu = D_0 + \mu A$ . Warning: We shall use the symbol  $\mu$  as fixed parameter and as function  $\mu(s)$ .
- (iii)  $\mu : \mathbb{R} \rightarrow \mathbb{R}$  is continuously differentiable, its derivative  $\mu'$  is bounded,  $\mu(0) = 1$  and  $\mu'(0) > 0$ . There exists  $s_i < 0$  and  $s_f > 0$  such that  $\mu(s) = 0$  if  $s < s_i$  or  $s > s_f$ .

**Remark 2.3** The condition above is fulfilled by a large class of critical potentials  $A$ . (i) the absence of resonances is generically fulfilled (see [15]). Assumption (ii) is not essential but makes the proof less heavy.

Under this condition (see e.g [41]) the operator of interest defining (2) namely  $\frac{1}{\varepsilon}D_{\mu(s)}$  generates a unitary time evolution denoted by  $U^\varepsilon(s, s_0)$  obeying

$$i\partial_s U^\varepsilon(s, s_0) = \frac{1}{\varepsilon}D_{\mu(s)}U^\varepsilon(s, s_0), \quad (5)$$

generating solutions of (2). The following theorem and its corollary assert that there exists a scattering solution of the Dirac equation (2) which at large negative times is element of the negative energy spectral subspace of the free Dirac operator ( $A = 0$ ) and at large positive times it is an element of the positive energy spectral subspace of the free Dirac operator.

**Theorem 2.4** (*Decay of Bound States*)

Assume Condition 2.2. Let  $\Phi_{\mu(s_0)}$  be a normalized bound state of  $D_{\mu(s_0)}$  for some  $s_0 \in (s_i, 0]$ . Let  $U^\varepsilon(s, s_0)$  be given by (5), i.e.  $\psi_s^\varepsilon = U^\varepsilon(s, s_0)\Phi_{\mu(s_0)}$  is the solution of the Dirac equation (2) with  $\psi_{s_0}^\varepsilon = \Phi_{\mu(s_0)}$ . Then for all  $\chi \in L^2$

$$\lim_{\varepsilon \rightarrow 0} \langle \psi_s^\varepsilon, \chi \rangle = 0 \quad (6)$$

for any  $s > 0$ .

As a corollary of Theorem 2.4 and the adiabatic theorem (see e.g. [40]) we obtain that the solution starts in the negative energy spectral subspace of the free Dirac-Hamiltonian  $D_0$  and ends in the positive energy spectral subspace. Denoting with  $P_0^+, P_0^-$  the corresponding spectral projectors we formulate

**Corollary 2.5** (*Adiabatic Pair Creation*) Let  $\psi_s^\varepsilon$  be as in Theorem 2.4. Then for all  $s < s_i$

$$\lim_{\varepsilon \rightarrow 0} \langle \psi_s^\varepsilon, P_0^- \psi_s^\varepsilon \rangle = 1. \quad (7)$$

and for all  $s > s_f$

$$\lim_{\varepsilon \rightarrow 0} \langle \psi_s^\varepsilon, P_0^+ \psi_s^\varepsilon \rangle = 1. \quad (8)$$

The proof of this consists in observing that (7) follows directly from the initial condition of  $\psi_s^\varepsilon$  (bound state in the gap) and the adiabatic theorem [39]. For (8) one must apply Theorem 2.4, which ensures that the scattering state is orthogonal to any bound state in the gap, then the statement follows again using the adiabatic theorem.

### 3 Skeleton of the Proof and Content of Sections

The proof of Theorem 2.4 consists in controlling the propagation of  $\psi_s^\varepsilon$ . For sufficiently small  $\varepsilon$ ,  $\psi_s^\varepsilon$  follows more or less the bound states  $\Phi_s$ . Reaching the critical time  $s = 0$  the bound state “vanishes” in the positive energy subspace of the free Dirac Hamiltonian. One needs to show that  $\psi_s^\varepsilon$  will stay there for all later times. Note that we need to control the wavefunction evolution for a time dependent hamiltonian.

The proof has thus naturally two parts:

- (1) Show that a bound state with energy in the energy gap between  $-1$  and  $1$  reaches adiabatically the upper spectral edge without “injuries” and ending up in  $\mathcal{N}$  (Lemma 7.1).
- (2) Show that any state in  $\mathcal{N}$  scatters during the time in which the potential stays overcritical (Corollary 7.7).

The proof of (1) is done in several sections. Contrary to what one might expect at first sight it is quite involved.

- (1.1) The problem is the possible degeneracy of the bound states. We must show that the eigenspaces  $\mathcal{N}_\mu$  of  $E_\mu$  when  $\mu$  goes to one converge to  $\mathcal{N}$  (Corollary 6.5). That is done in the first part of Section 6. The proof is long and tricky and aims at the definition of an invertible operator  $Q_\mu$  which maps  $\mathcal{N}_\mu$  to  $\mathcal{N}$ . This map will be used in the next step.

- (1.2) We need good control of “how a bound state converges”. We show that to every bound state  $\Phi \in \mathcal{N}$  the sequence of bound states  $(\Phi_\mu)_\mu \subset \mathcal{N}_\mu$  given by  $\Phi_\mu = Q_\mu^{-1}\Phi$  is “good” in the sense that differentiable with respect to  $\mu$  with relatively soft divergent behavior of the derivative when  $\mu$  tends to one (Lemma 6.6). This is done in section 6.1.
- (1.3) The “good sequences”  $(\Phi_\mu)_\mu$  will be used in Section 7.1 to establish an adiabatic lemma without a gap, Lemma 7.1. The proof is a two scale argument. We first propagate adiabatically to times  $s_0$  very close to 0 and then by the uniform (in  $\varepsilon$ ) estimate (69) we can close the gap. This establishes the first part of the proof.

The proof of (2) is naturally much more involved than that of (1).

- (2.1) We wish to show that any bound state at the spectral edge scatters. What we need to establish is that the wavefunction leaves the range of the scattering potential sufficiently fast, faster than  $\varepsilon^{-1}$ , the time after which the potential is undercritical again. Such control is rather easy when the potential is not depending on time, but here it depends on time: How does one control the evolution of wavefunctions for time dependent hamiltonians? The most direct and physical way is using generalized eigenfunctions. The point is however that the generalized eigenfunctions are singular near the spectral edge! The essential question is: How bad? We recall here and rely heavily on a result of [22, 21] on generalized eigenfunctions near criticality (Theorem 4.2). That is done in Section 4.
- (2.2) The eigenfunctions allow us to control the wavefunction evolution for potentials constant in time. Section 5 gives with Lemma 5.1 preliminary estimates. (23) shows what we need to have when dealing with generalized eigenfunctions namely an estimate in the sup-norm. The proof of Lemma 5.1 involves tricky use of momentum cut-offs. Corollary 5.2 formulates then three estimates for the  $L^2$ - norms. At first sight the third estimate ((iii) of Corollary 5.2) seems the only relevant one, but the first two are in fact needed for technical reasons later. (The technical reason is that we must bring the “static potential estimates” in contact with the true time evolution, i.e. with the non-static potential). The proofs here are “straightforward” applications of stationary phase methods, taking however the singular eigenfunctions behavior

into account by tricky cut-offs of small momenta. The stationary phase application leading to good decay is unfortunately lengthy, while not difficult nor tricky. Therefore we decided to shift that calculation to the Appendix.

- (2.3) In section 7.2 the contact with the true time evolution is made. First we consider the wavefunction evolution for “short times” (Lemma 7.3). Short means macroscopic times of order one. We introduce the time  $\sigma$  which may roughly be thought of as being the first time at which  $\mu$  reaches a maximum. Here we use the estimate (iii) of Corollary 7.2, which is the translation of Corollary 5.2 to the macroscopic time scale. It is shown that most of the wavefunction will have left the range of the potential by a macroscopic time of order  $\varepsilon^{1/3}$ , i.e.  $\varepsilon^{-2/3}$  on the microscopic time scale. That proof uses Cook’s argument in combination with physical insights: We need to compare the true evolution until time  $s \leq \sigma$  with a “static potential” evolution. The potential will be “frozen” to the value it has at a time  $s$ . There is a big error between the true and the auxiliary time evolution. But in terms of the evolution of the relevant part of the wavefunction the error is not so big, since most of the wavefunction will have left the range of the potential. So the error is transported to a region in space which we do not care so much about. That idea is behind this part of the proof. Of course, we must insure that for very long times, this error does not come back! But that is done in the next step.
- (2.4) In Section 7.3 we extend our result to all times (Lemma 7.3). Most of the wavefunction has left the range of the potential, we must insure that it stays like that. Again we use Cook’s method, but now we freeze the potential at the value it has at time  $\sigma$ . In fact we can chose here any value  $s > 0$  for which the potential is overcritical. The physical idea is clear: Since most of the wavefunction is outside of the range of the potential it moves freely and the critical potential is roughly the zero potential. To avoid problems arising from small  $k$  values, we use density arguments and cut off small momenta. It is here where Corollary 7.2 (i) and (ii) come into play.
- (2.5) Section 7.4 collects simply the results to establish the second part of Theorem 2.4, namely that the bound state which reached the upper edge scatters.

## 4 Generalized Eigenfunctions

Consider for  $\mu \in \mathbb{R}$ ,  $\mathbf{k} \in \mathbb{R}^3$ ,  $j = 1, 2, 3, 4$  and  $E_k = \sqrt{k^2 + 1}$  the bounded classical solutions  $\varphi_\mu(\mathbf{k}, j, \mathbf{x})$  (generalized eigenfunctions (GEF)) given by

$$D_\mu \varphi_\mu(\mathbf{k}, j, \cdot) = \pm E_k \varphi_\mu(\mathbf{k}, j, \cdot), \quad (9)$$

with “+” for  $j = 1, 2$  and “-” for  $j = 3, 4$ .

**Lemma 4.1** (*GEF Properties*)

Let  $A$  satisfy condition 2.2 (i) and  $\delta > 0$  be such, that  $\mu = 1$  is the only critical value in  $[1, 1 + \delta]$ . Then

- (a) for any  $\mathbf{k} \in \mathbb{R}^3$ , any  $j = 1, \dots, 4$  and any  $\mu \in (1, 1 + \delta]$  there exists (up to linearity) a unique solution  $\varphi_\mu(\mathbf{k}, j, \cdot)$  of (9).
- (b) for all  $\mathbf{x}$  the functions  $\varphi_\mu(\mathbf{k}, j, \mathbf{x})$  are infinitely often continuously differentiable with respect to  $k$  for  $\mathbf{k} \neq 0$ .
- (c) The scattering system  $(D_0, D_\mu = D_0 + \mu A)$  is asymptotically complete for any  $\mu \in [1, 1 + \delta]$ . In particular the wave operator  $\Omega_\mu^+$  defined via

$$\Omega_\mu^+ \psi \equiv \lim_{t \rightarrow \infty} e^{iD_\mu t} e^{-iD_0 t} \psi \quad \text{for all } \psi \in L^2$$

exists, is isometric and

$$\text{Ran } \Omega_\mu^+ = \mathcal{H}_\mu^{\text{ac}},$$

where  $\mathcal{H}_\mu^{\text{ac}}$  is the spectral subspace of the absolutely continuous spectrum of  $D_\mu$ .

- (d) for  $\mu \neq 1$  the solutions  $\varphi_\mu(\mathbf{k}, j, \cdot)$  define a generalized Fourier transform, i.e. an isometry  $\mathcal{F} : \mathcal{H}_\mu^{\text{ac}} \subset L^2(\mathbb{R}^3, \mathbb{C}^4) \rightarrow L^2((\mathbb{R}^3, \{1, 2, 3, 4\}), \mathbb{C})$  by

$$\mathcal{F}_\mu(\psi)(\mathbf{k}, j) := (2\pi)^{-\frac{3}{2}} \int \overline{\varphi}_\mu(\mathbf{k}, j, \mathbf{x}) \psi(\mathbf{x}) d^3x \quad (10)$$

and

$$\psi(\mathbf{x}) = \sum_{j=1}^4 \int (2\pi)^{-\frac{3}{2}} \varphi_\mu(\mathbf{k}, j, \mathbf{x}) \mathcal{F}_\mu(\psi)(\mathbf{k}, j) d^3k, \quad (11)$$

where the integrals are in the l.i.m.-sense (see e.g. [28]). Furthermore Plancherel holds

$$\langle \psi, \chi \rangle = \sum_{j=1}^4 \int \mathcal{F}_\mu(\psi)^*(\mathbf{k}, j) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) d^3k \quad (12)$$

as well as Parseval

$$\|\psi\|^2 = \sum_{j=1}^4 \int |\mathcal{F}_\mu(\psi)(\mathbf{k}, j)|^2 d^3k =: \|\mathcal{F}_\mu(\psi)\|^2. \quad (13)$$

**Proof:** (a) and (b) have been proven in [7] (see Lemma 3.4. therein). Also (c) is not new, it is known to hold for short range potentials (see for example Theorems 8.2, 8.3 and 8.20 in [41]). (d) follows also from Lemma 3.4. in [7], where it is shown that  $\mathcal{F}_\mu(\psi)(\mathbf{k}, j) = \widehat{\psi}_\mu^{out}$  where the  $\widehat{\cdot}$  stands for the (ordinary) Fourier transform and  $\psi_\mu^{out}$  is defined by  $\Omega_\mu^+ \psi_\mu^{out} = \psi$ .

Since  $\Omega_\mu^+$  is isometric we have

$$\begin{aligned} \langle \psi, \chi \rangle &= \langle \Omega_\mu^+ \psi, \Omega_\mu^+ \chi \rangle = \langle \psi_\mu^{out}, \chi_\mu^{out} \rangle = \sum_{j=1}^4 \int \widehat{\psi}_\mu^{out*}(\mathbf{k}, j) \widehat{\chi}_\mu^{out}(\mathbf{k}, j) d^3k \\ &= \sum_{j=1}^4 \int \mathcal{F}_\mu(\psi)^*(\mathbf{k}, j) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) d^3k, \end{aligned}$$

i.e. (12) and (13) hold.  $\square$

As we deal with a time dependent external field which grows from under-criticality to over-criticality in the course of which we need very good control on the evolution of the wavefunction, we need *uniform* estimates on the generalized eigenfunctions of the operator  $D_0 + \mu A$ . Uniform estimates have not been given before. It is known, that for critical potentials the generalized eigenfunctions diverge for  $k \rightarrow 0$  [14], but that is not sufficient to control the propagation. What is sufficient are estimates on the  $L^\infty$ -norm of the generalized eigenfunctions of  $D_0 + \mu A$  and their  $k$ -derivatives *uniform* in  $\mathbf{k}$  and *uniform* in  $\mu \in [1 - \delta, 1 + \delta]$  for some  $\delta > 0$ . The uniform estimates we need are provided in [22]. We cite the crucial Corollaries 3.7. and 6.3. in [22], using it (following Remark 3.6. in [22]) both for positive and negative energies, i.e. for spin indices  $j = 1, 2$  and  $j = 3, 4$ . Assuming that there is no bound state nor resonance with energy  $-1$  we could give a much better

bound for negative energies, but that is not necessary for our estimates below where the bounds on  $j = 1, 2$  are essential.

**Theorem 4.2** (*Upper Bound for the Sup-Norm of the GEF of Lemma 4.1*)  
There exist  $\delta > 0$ , constants  $\nu_l$ ,  $1 \leq l \leq n$  and a constant  $c > 0$  so that the following holds: For the  $m^{\text{th}}$  derivative  $\varphi_\mu^{(m)} := \partial_k^m \varphi_\mu$ ,  $m \in \mathbb{N}_0$ , there exist constants  $C_m$  so that for every  $\mathbf{k} \in \mathbb{R}^3$ ,  $j = 1, 2, 3, 4$  and for every  $\mu \in [1 - \delta, 1 + \delta]$

$$\|(1 + |\cdot|)^{-m} \varphi_\mu^{(m)}(\mathbf{k}, j, \cdot)\|_\infty < C_m \left( k^{-m} + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right|^{m+1} \right). \quad (14)$$

Furthermore there exist  $\Xi_\mu(\mathbf{k}, j, \cdot) \in \mathcal{N}$  and  $C$  uniform in  $\mathbf{k} \in \mathbb{R}^3$  and  $\mu \in [1 - \delta, 1 + \delta]$  so that

$$\|\varphi_\mu(\mathbf{k}, j, \cdot) - \Xi_\mu(\mathbf{k}, j, \cdot)\|_\infty < C. \quad (15)$$

## 5 Propagation Estimates

In this section we want to apply Theorem 4.2 to get estimates on the time propagation of wave-functions for the static Dirac Hamiltonian  $D_\mu$  uniform in  $\mu \in [1 - \delta, 1 + \delta]$ .

Under Condition 2.2 (see e.g. [41])  $D_\mu$  generates a unitary time evolution denoted by  $V_\mu(t, 0)$ , i.e.

$$i\partial_t V_\mu(t, 0) = D_\mu V_\mu(t, 0), \quad (16)$$

which applied to eigenfunctions reads

$$V_\mu(t, 0)\varphi_\mu(\mathbf{k}, j, \mathbf{x}) = e^{-iE_k t} \varphi_\mu(\mathbf{k}, j, \mathbf{x}). \quad (17)$$

This formula explains the role of the generalized eigenfunctions and it gives us the most direct control on the evolution of wavefunctions by expanding the wavefunction into generalized eigenfunctions. The estimates we are after are such that we can control the wavefunction evolution of the bound states in  $\mathcal{N}$  during overcriticality. The bound states must decay fast enough (i.e. scatter fast enough) so that they are outside of the range  $\mathcal{S}_A$  of the potential before the potential becomes undercritical again. The naive picture of scattering

theory suggests that the Fourier transform (given by plane waves) of the state governs the spreading. But we are here in a delicate situation analogous to resonant behavior. The generalized eigenfunctions are not at all like plane waves as we see from (14) and we must control the spreading given by such badly behaved eigenfunctions. We need to separate very very slow spreadings of the wavefunction (whose contribution will be hopefully negligible because of small probability) from the moderately fast spreading which make the state scatter. The borderline will be given by the  $k$ -value of the “resonance”, i.e., where  $\mu - 1 \approx \nu_l k^2$  (c.f. (14)).  $\mu$  should be thought of being only slightly bigger than 1, because that is the dangerous regime, the regime where  $k$  is small with large probability. For technical reasons we also separate very large momenta. Thus we will give propagation estimates for a wavefunction separating large, intermediate and small momenta, using the mollifier  $\widehat{\rho}_\kappa \in C^\infty$  given by

$$\widehat{\rho}(\mathbf{k}) := \begin{cases} 0, & \text{for } k \leq 1, \\ 1, & \text{for } k \geq 2, \end{cases} \quad (18)$$

and for  $\kappa > 0$  we define

$$\widehat{\rho}_\kappa(\mathbf{k}) := \widehat{\rho}\left(\frac{\mathbf{k}}{\kappa}\right) \quad (19)$$

and the operator

$$\rho_{\kappa,\mu} := \mathcal{F}_\mu^{-1} \widehat{\rho}_\kappa \mathcal{F}_\mu. \quad (20)$$

Note that  $[D_\mu, \rho_{\kappa,\mu}] = 0$ .

**Lemma 5.1** (*Cutoff and propagation estimates - stationary case*)

Let  $\delta > 0$  be such that there is no bound state of  $D_\mu$  for  $\mu \in (1, 1 + \delta]$ , i.e.  $\mathcal{H}^{ac}(D_\mu) = L^2$ . Let  $\mathcal{S} \subset \mathbb{R}^3$  be compact. Then there exists a  $C < \infty$  such that for any  $\chi \in L^2$  with  $\text{supp}\chi \subset \mathcal{S}$  and any  $0 < \underline{\kappa} < 1$  we have that for all  $\mu \in (1, 1 + \delta]$  and all  $0 \leq t < \infty$

$$\|(1 - \rho_{\underline{\kappa},\mu})\chi\| \leq C \underline{\kappa}^{\frac{3}{2}} \sup_{k < 2\underline{\kappa}} \left( 1 + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right) \|\chi\| \quad (21)$$

and

$$\|V_\mu(t, 0)(1 - \rho_{\underline{\kappa},\mu})\chi\|_\infty \leq C \underline{\kappa}^3 \sup_{k < 2\underline{\kappa}} \left( 1 + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right)^2 \|\chi\|. \quad (22)$$



Furthermore let  $\bar{\kappa} < \infty$ . For all  $m \in \mathbb{N}_0$  there exists  $C_m < \infty$  such that

$$\begin{aligned} & \|\mathbb{1}_{\mathcal{S}} V_\mu(t, 0) \rho_{\underline{\kappa}, \mu} (1 - \rho_{\bar{\kappa}, \mu}) \chi\|_\infty \\ & < \bar{\kappa}^3 t^{-m} C_m \underline{\kappa}^{-4} \left( \underline{\kappa}^{-2} + \sup_{2\bar{\kappa} > k > \underline{\kappa}} \left| \sum_{l=1}^n \frac{1}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right)^m \|\chi\| \end{aligned} \quad (23)$$

and

$$\|\rho_{\underline{\kappa}, \mu} \rho_{\bar{\kappa}, \mu} \chi\| \leq \frac{\|D_\mu \chi\|}{\bar{\kappa}}. \quad (24)$$

**Proof:** Using linearity we can assume without loss of generality that  $\|\chi\| = 1$ . We start with (21). Let  $\chi$  be as in the Lemma. Then by (20) and (13)

$$\|(1 - \rho_{\underline{\kappa}, \mu}) \chi\| = \|(1 - \hat{\rho}_{\underline{\kappa}}) \mathcal{F}_\mu(\chi)\| \leq 4 \sup_{j, k < 2\underline{\kappa}} \{|\mathcal{F}_\mu(\chi)(\mathbf{k}, j)|\} \|1 - \hat{\rho}_{\underline{\kappa}}\| \quad (25)$$

By (19) and (18)

$$\|\hat{\rho}_{\underline{\kappa}} - 1\| = \underline{\kappa}^{\frac{3}{2}} \left( \int |\hat{\rho}(p) - 1|^2 d^3 p \right)^{\frac{1}{2}} \leq C \underline{\kappa}^{3/2}. \quad (26)$$

Furthermore

$$\begin{aligned} \sup_{j, k < 2\underline{\kappa}} \{|\mathcal{F}_\mu(\chi)(\mathbf{k}, j)|\} & \leq \sup_{j, k < 2\underline{\kappa}} \left\{ \int (2\pi)^{-\frac{3}{2}} |\bar{\varphi}_\mu(\mathbf{k}, j, \mathbf{x}) \chi(\mathbf{x})| d^3 x \right\} \\ & \leq \sup_{j, k < 2\underline{\kappa}} \{ \|\varphi_\mu(\mathbf{k}, j, \cdot)\|_\infty \} (2\pi)^{-\frac{3}{2}} \|\chi\|_1, \end{aligned} \quad (27)$$

where by Schwartz

$$\begin{aligned} \|\chi\|_1 & = \left| \int |\chi(\mathbf{x})| d^3 x \right| \\ & = \left| \int \mathbb{1}_{\mathcal{S}} |\chi(\mathbf{x})| d^3 x \right| \\ & \leq \|\chi(\mathbf{x})\| \sqrt{|\mathcal{S}|} \leq C. \end{aligned} \quad (28)$$

For  $\sup_{j, k < 2\underline{\kappa}} \{ \|\varphi_\mu(\mathbf{k}, j, \cdot)\|_\infty \}$  we have by Theorem 4.2 (14) for  $m = 0$  that

$$\sup_{j, k < 2\underline{\kappa}} \{ \|\varphi_\mu(\mathbf{k}, j, \cdot)\|_\infty \} \leq C \sup_{k < 2\underline{\kappa}} \left( 1 + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right). \quad (29)$$

Hence for (27) with (28)

$$\sup_{j,k < 2\underline{\kappa}} \{ | \mathcal{F}_\mu(\chi)(\mathbf{k}, j) | \} \leq C \sup_{k < 2\underline{\kappa}} \left( 1 + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right). \quad (30)$$

This with (26) in (25) yields (21).

Next we establish (22). Observing the definitions

$$\begin{aligned} & \|V_\mu(t, 0)(1 - \rho_{\underline{\kappa}, \mu})\chi\|_\infty \\ & \leq \sum_{j=1}^4 \left\| \int (2\pi)^{-\frac{3}{2}} | V_\mu(t, 0)\varphi_\mu(\mathbf{k}, j, \mathbf{x})\mathcal{F}_\mu(\chi)(\mathbf{k}, j)(1 - \widehat{\rho}_{\underline{\kappa}}(\mathbf{k})) | d^3k \right\|_\infty \\ & = \sum_{j=1}^4 \left\| \int (2\pi)^{-\frac{3}{2}} | e^{-iE_k t}\varphi_\mu(\mathbf{k}, j, \mathbf{x})\mathcal{F}_\mu(\chi)(\mathbf{k}, j)(1 - \widehat{\rho}_{\underline{\kappa}}(\mathbf{k})) | d^3k \right\|_\infty \\ & \leq 4 \sup_{j,k < 2\underline{\kappa}} \{ \|\varphi_\mu(\mathbf{k}, j, \cdot)\|_\infty \} \sup_{j,k < 2\underline{\kappa}} \{ | \mathcal{F}_\mu(\chi)(\mathbf{k}, j) | \} \|1 - \widehat{\rho}_{\underline{\kappa}}\|_1. \end{aligned}$$

Similarly as in (26) we find that  $\|1 - \widehat{\rho}_{\underline{\kappa}}\|_1 < C\underline{\kappa}^3$  and with (29) and (30) we get (22).

We now turn to (23). As above

$$\begin{aligned} & V_\mu(t, 0)\rho_{\underline{\kappa}, \mu}(1 - \rho_{\overline{\kappa}, \mu})\chi(\mathbf{x}) \\ & = \sum_{j=1}^4 \int (2\pi)^{-\frac{3}{2}} V_\mu(t, 0)\varphi_\mu(\mathbf{k}, j, \mathbf{x})\widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}})\mathcal{F}_\mu(\chi)(\mathbf{k}, j)d^3k \\ & = \sum_{j=1}^4 \int (2\pi)^{-\frac{3}{2}} \exp(-itE_k)\varphi_\mu(\mathbf{k}, j, \mathbf{x})\widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}})\mathcal{F}_\mu(\chi)(\mathbf{k}, j)d^3k. \end{aligned} \quad (31)$$

It is for this term that we need the behavior of the derivatives of the eigenfunctions (c.f. (14)). We shall use a stationary phase method, using  $\frac{iE_k}{kt}\partial_k \exp(-itE_k) = \exp(-itE_k)$ . The rigorous estimate of this formula is based on a simple straightforward computation which is done in the appendix. We shall only describe here in a heuristic manner how the estimate comes about.

First we recall (14)

$$\|(1+x)^{-m}\varphi_\mu^{(m)}(\mathbf{k}, j, \cdot)\|_\infty \leq C_m \left( k^{-m} + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right|^{m+1} \right). \quad (32)$$

This enters also in the  $k$ -derivatives of  $\mathcal{F}_\mu(\chi)$ . Since  $\chi$  has compact support in  $\mathcal{S}$  we obtain

$$\begin{aligned} |\partial_k^m \mathcal{F}_\mu(\chi)(\mathbf{k}, j)| &= |\partial_k^m \langle \varphi_\mu(\mathbf{k}, j, \cdot), \chi \rangle| \\ &\leq C_m \|(1+x)^{-m} \varphi_\mu^{(m)}(\mathbf{k}, j, \cdot)\|_\infty \|(1+x)^m \chi\|_1 \\ &\leq CC_m \|(1+x)^{-m} \varphi_\mu^{(m)}(\mathbf{k}, j, \cdot)\|_\infty \end{aligned} \quad (33)$$

where in the last step we followed (28). Doing the partial integration we need to apply the operator  $\partial_k \frac{E_k}{k} = \frac{1}{k^2} + \frac{E_k}{k} \partial_k$ . Relevant to us is only the ‘‘small’’  $k$ -behavior ( $k \geq \underline{\kappa}$ ), i.e. we need to count the inverse powers of  $k$ . In that sense  $\partial_k \frac{E_k}{k} \sim \frac{1}{k^2} + \frac{1}{k} \partial_k$ . Further observe that the relevant term in (31) to which  $\partial_k \frac{E_k}{k}$  is applied is the product  $\varphi_\mu(\mathbf{k}, j, \mathbf{x}) \widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}}) \mathcal{F}_\mu(\chi)(\mathbf{k}, j)$ . But  $\partial_k \widehat{\rho}_{\underline{\kappa}} \sim \frac{1}{\underline{\kappa}} \sim \frac{1}{k}$ , while

$$\begin{aligned} \partial_k \varphi_\mu^{(m)}(\mathbf{k}, j, \mathbf{x}) &= \varphi_\mu^{(m+1)}(\mathbf{k}, j, \mathbf{x}) \\ &\approx (1+x) \varphi_\mu^{(m)}(\mathbf{k}, j, \mathbf{x}) \left( k^{-1} + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right). \end{aligned}$$

Likewise for  $\mathcal{F}_\mu(\chi)$ . Since we are interested in the supremum over the compact set  $\mathcal{S}$  the factor  $(1+x)$  can be estimated by a constant. The upshot is then that the contribution of the terms is essentially the  $m$ -th power of

$$\begin{aligned} \partial_k \frac{E_k}{k} &\approx \frac{1}{k^2} + \frac{1}{k} \left( k^{-1} + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right) \\ &\approx \frac{1}{k^2} + \left| \sum_{l=1}^n \frac{1}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \end{aligned}$$

multiplied by the product  $\varphi_\mu \mathcal{F}_\mu(\chi)$ , yielding roughly

$$\left( \frac{1}{k^2} + \left| \sum_{l=1}^n \frac{1}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right)^m \left( 1 + \left| \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right| \right)^2.$$

The second factor can be bounded by  $c^{-2} k^{-4} \leq C \underline{\kappa}^{-4}$ . This gives with the volume factor  $\overline{\kappa}^3$  the right hand side of (23).

Finally we establish (24). By (13) and using  $E_k^2 \geq k^2$

$$\begin{aligned} \|\rho_{\underline{\kappa}, \mu} \rho_{\overline{\kappa}, \mu} \chi\|^2 &\leq \int_{k > \overline{\kappa}} |\mathcal{F}_\mu(\chi)|^2 d^3 k \leq \int_{k > \overline{\kappa}} E_k^{-2} \mathcal{F}_\mu(D_\mu \chi)^* \mathcal{F}_\mu(D_\mu \chi) d^3 k \\ &\leq \frac{1}{\overline{\kappa}^2} \int_{k > \overline{\kappa}} \mathcal{F}_\mu(D_\mu \chi)^* \mathcal{F}_\mu(D_\mu \chi) d^3 k = \frac{\|D_\mu \chi\|^2}{\overline{\kappa}^2}. \end{aligned}$$

□

The results of Lemma 5.1 can now be used to estimate the decay behavior of any compactly supported wavefunction  $\chi \in L^2$

**Corollary 5.2** (*Propagation Estimate - stationary case*)

Let  $\mathcal{S} \subset \mathbb{R}^3$  be compact,  $0 < \xi < 1$ . There exists a  $\delta > 0$  (possibly smaller than the  $\delta$  of Lemma 5.1) such that there exists a  $C < \infty$  and for all  $\tilde{m} \in \mathbb{N}$  a constants  $C_{\tilde{m}}$  such that for all  $\mu \in (1, 1 + \delta]$ , all  $t > (\mu - 1)^{-\frac{3}{2(1-\xi)}}$  and all  $\chi \in L^2$  with  $\text{supp}\chi \subset \mathcal{S}$  the following holds:

(i) Let  $V_\mu$  be defined by (16), then for  $\underline{\kappa} = t^{-\frac{1}{2}(1-\xi)}$  and for all  $w \geq t$

$$\|\mathbb{1}_{\mathcal{S}}V_\mu(w, 0)\rho_{\underline{\kappa}, \mu}\chi\| \leq C_{\tilde{m}}w^{-\tilde{m}}(\|D_\mu\chi\|),$$

(ii) for  $\underline{\kappa} = t^{-\frac{1}{2}(1-\xi)}$

$$\|(1 - \rho_{\underline{\kappa}, \mu})\chi\| \leq Ct^{-\frac{3}{4}(1-\xi)}(\mu - 1)^{-\frac{1}{4}}\|\chi\|,$$

(iii)

$$\|\mathbb{1}_{\mathcal{S}}V_\mu(t, 0)\chi\| \leq C|\mu - 1|^{-\frac{1}{2}}t^{-\frac{3}{2}(1-\xi)}(\|\chi\| + \|D_\mu\chi\|).$$

We note that we want to have good decay estimates, so  $\xi$  should be thought of being small. We also wish to remark that we shall need both estimates (i) and (ii) as well as (iii). (iii) is better than (i) and (ii) together but it is not suitable for “density arguments” which we shall use later on when we compare the true time evolution with the  $V_\mu$ -evolution.

**Proof:** Using linearity we can assume without loss of generality that  $\|\chi\| = 1$ . We start with (ii). With our choice  $\underline{\kappa} = t^{-\frac{1}{2}(1-\xi)}$  we obtain in view of (21)

$$\|(1 - \rho_{\underline{\kappa}, \mu})\chi\| \leq Ct^{-\frac{3}{4}(1-\xi)} \sup_{k < 2\underline{\kappa}} \left( 1 + \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right).$$

Since by assumption  $t > (\mu - 1)^{-\frac{3}{2(1-\xi)}}$  we have  $\underline{\kappa}^2 < (\mu - 1)^{3/2} \ll (\mu - 1)$  and hence for  $\delta$  small enough we are below the resonant  $k$ -values, i.e.  $\inf_l |\mu -$

$1 - \nu_l k^2 \geq (\mu - 1)/2$  and thus the supremum of the bracket term is less than  $(1 + C_{\underline{\kappa}}/(\mu - 1))$ . Thus

$$\begin{aligned} \|(1 - \rho_{\underline{\kappa}, \mu})\chi\| &\leq C t^{-\frac{3}{4}(1-\xi)} \left(1 + \frac{C_{\underline{\kappa}}}{\mu - 1}\right) \\ &\leq C t^{-\frac{3}{4}(1-\xi)} \left(1 + \frac{C(\mu - 1)^{3/4}}{\mu - 1}\right) \\ &\leq C t^{-\frac{3}{4}(1-\xi)} (\mu - 1)^{-1/4}, \end{aligned}$$

which establishes (ii).

We now prove (i). Using

$$\|\mathbf{1}_{\mathcal{S}}\psi\| \leq \|\mathbf{1}_{\mathcal{S}}\| \|\mathbf{1}_{\mathcal{S}}\psi\|_{\infty} \leq C \|\mathbf{1}_{\mathcal{S}}\psi\|_{\infty} \quad (34)$$

we have with the high momentum cutoff  $\bar{\kappa}$  to be specified below

$$\begin{aligned} &\|\mathbf{1}_{\mathcal{S}}V_{\mu}(w, 0)\rho_{\underline{\kappa}, \mu}\chi\| \\ &\leq \|\mathbf{1}_{\mathcal{S}}V_{\mu}(w, 0)\rho_{\underline{\kappa}, \mu}(1 - \rho_{\bar{\kappa}, \mu})\chi\| + \|\mathbf{1}_{\mathcal{S}}V_{\mu}(w, 0)\rho_{\underline{\kappa}, \mu}\rho_{\bar{\kappa}, \mu}\chi\| \\ &\leq C \|\mathbf{1}_{\mathcal{S}}V_{\mu}(w, 0)\rho_{\underline{\kappa}, \mu}(1 - \rho_{\bar{\kappa}, \mu})\chi\|_{\infty} + \|\rho_{\underline{\kappa}, \mu}\rho_{\bar{\kappa}, \mu}\chi\|. \end{aligned} \quad (35)$$

Let  $w \geq t > (\mu - 1)^{-\frac{3}{2(1-\xi)}}$ .  $\bar{\kappa}$  will be chosen large enough, so that the first term encompasses the resonant regime ( $\nu_l k^2 \approx \mu - 1$ ). We obtain with Lemma 5.1 formula (23) using  $w \geq t > (\mu - 1)^{-\frac{3}{2(1-\xi)}}$  and  $\underline{\kappa} = t^{-\frac{1}{2}(1-\xi)}$

$$\begin{aligned} \|\mathbf{1}_{\mathcal{S}}V_{\mu}(w, 0)\rho_{\underline{\kappa}, \mu}(1 - \rho_{\bar{\kappa}, \mu})\chi\|_{\infty} &\leq C C_m \bar{\kappa}^3 w^{-m} t^{2-2\xi} \left(t^{1-\xi} + C(\mu - 1)^{-\frac{3}{2}}\right)^m \\ &\leq C C_m \bar{\kappa}^3 w^{-m} (C t^{1-\xi})^{m+2} \\ &\leq \tilde{C}_m \bar{\kappa}^3 w^{-m\xi+2-2\xi}. \end{aligned}$$

For the second term in (35) we get with Lemma 5.1 formula (24)

$$\|\rho_{\underline{\kappa}, \mu}\rho_{\bar{\kappa}, \mu}\chi\| \leq \frac{\|D_{\mu}\chi\|}{\bar{\kappa}}.$$

Hence for (35) we obtain

$$\|\mathbf{1}_{\mathcal{S}}V_{\mu}(w, 0)\rho_{\underline{\kappa}, \mu}\chi\| \leq \tilde{C}_m \bar{\kappa}^3 w^{-m\xi+2-2\xi} + \frac{\|D_{\mu}\chi\|}{\bar{\kappa}}.$$

Choosing  $\bar{\kappa} := w^{m\xi/4}$  we find the bound

$$\|\mathbf{1}_S V_\mu(w, 0) \rho_{\underline{\kappa}, \mu} \chi\| \leq \tilde{C}_m \|D_\mu \chi\| w^{-\frac{m\xi}{4} + 2 - 2\xi}.$$

Choosing  $m$  such that  $m\xi/4 - 2 + 2\xi > \tilde{m}$  (i) follows.

Next we prove (iii). By (34)

$$\begin{aligned} \|\mathbf{1}_S V_\mu(t, 0) \chi\| &\leq \|\mathbf{1}_S V_\mu(t, 0) \rho_{\underline{\kappa}, \mu} \chi\| + \|\mathbf{1}_S V_\mu(t, 0) (1 - \rho_{\underline{\kappa}, \mu}) \chi\|_\infty \\ &\leq \|\mathbf{1}_S V_\mu(t, 0) \rho_{\underline{\kappa}, \mu} \chi\| + \|V_\mu(t, 0) (1 - \rho_{\underline{\kappa}, \mu}) \chi\|_\infty. \end{aligned}$$

For the first summand use (i) with  $w = t$  and  $\tilde{m} = 2$ . For the second summand use Lemma 5.1 formula (22) with  $\underline{\kappa} = t^{-\frac{1}{2}(1-\xi)}$  to obtain

$$\begin{aligned} \|V_\mu(t, 0) (1 - \rho_{\underline{\kappa}, \mu}) \chi\|_\infty &\leq C t^{-\frac{3}{2}(1-\xi)} \sup_{k < 2\underline{\kappa}} \left( 1 + \sum_{l=1}^n \frac{k}{|\mu - 1 - \nu_l k^2| + ck^3} \right)^2 \\ &\leq C |\mu - 1|^{-\frac{1}{2}} t^{-\frac{3}{2}(1-\xi)}, \end{aligned}$$

where the bound comes as in the proof of (ii) with the only difference being that we now have the square.

□

## 6 $\mu$ -Convergence of the Eigenspaces

Let  $[\mu_B, 1]$  be a interval of parameters for which bound states for

$$D_\mu \Phi_\mu := (D_0 + \mu A) \Phi_\mu = E_\mu \Phi_\mu$$

exist. In the course of this paper we shall adjust  $\mu_B < 1$  according to our needs. Note that  $E_\mu \in [-1, 1]$ . Let  $\mathcal{N}_\mu$  denote the eigenspace of  $E_\mu$ . In [15] it is shown that Condition 2.2 (i) implies that there exist constants  $\mu_B < 1$ ,  $\underline{C} > 0$  and  $\overline{C} > 0$  such that

$$\underline{C} < \partial_\mu E_\mu < \overline{C} \tag{36}$$

for all  $\mu_B \leq \mu \leq 1$ .

**Lemma 6.1** *Let  $A$  satisfy Condition 2.2. Then for  $\mu_B$  close enough to 1*

$$\dim \mathcal{N} = \dim \mathcal{N}_\mu$$

for all  $\mu \in [\mu_B, 1)$ .

**Proof:** We shall present a proof which prepares notation and results which we shall need later on for the control of the  $\mu$ -derivative of the bound states: Given a critical degenerate bound state  $\Phi$  there is a special sequence of undercritical bound states converging to  $\Phi$  in a “nicely” differentiable manner. This special sequence will be called a “good sequence corresponding to  $\Phi$ ”. This notion is essential for the proof of the Adiabatic Lemma without a gap. The good sequence will be constructed via a “good operator”  $Q_\mu$ , which will be a bijection mapping  $\mathcal{N}_\mu$  to  $\mathcal{N}$ . The inverse of  $Q_\mu$  will be constructed from the resolvent of an operator  $R_\mu$  which we construct first. So before we prove the lemma we shall be concerned with  $R_\mu$  and  $Q_\mu$  the upshot of which is Corollary 6.3. Having that the actual proof of the lemma is very short. Let  $P_{\mathcal{N}}, P_{\mathcal{N}_\mu}$  be the projections on the eigen-spaces  $\mathcal{N}, \mathcal{N}_\mu$ . With

$$(D_\mu - E_\mu)P_{\mathcal{N}_\mu} = 0 \quad (37)$$

and

$$D_\mu - D_\nu = (\mu - \nu)A \quad (38)$$

we have

$$(D_1 - E_\mu)P_{\mathcal{N}_\mu} = (1 - \mu)AP_{\mathcal{N}_\mu} . \quad (39)$$

Since for  $\mu \in [\mu_B, 1)$   $E_\mu$  is in the resolvent set of  $D_1$

$$P_{\mathcal{N}_\mu} = (1 - \mu)(D_1 - E_\mu)^{-1}AP_{\mathcal{N}_\mu} . \quad (40)$$

Multiplying with  $P_{\mathcal{N}} - \mathbf{1} = -P_{\mathcal{N}^\perp}$  and observing that  $[D_1, P_{\mathcal{N}^\perp}] = 0$  yields

$$(P_{\mathcal{N}} - \mathbf{1})P_{\mathcal{N}_\mu} = -(1 - \mu)(D_1 - E_\mu)^{-1}P_{\mathcal{N}^\perp}AP_{\mathcal{N}_\mu} ,$$

thus

$$P_{\mathcal{N}}P_{\mathcal{N}_\mu} = (1 - (1 - \mu)(D_1 - E_\mu)^{-1}P_{\mathcal{N}^\perp}A)P_{\mathcal{N}_\mu} . \quad (41)$$

Note that  $(D_1 - E_\mu)^{-1}$  is now applied to vectors orthogonal to  $\mathcal{N}$ . Therefore the term has good chances of being controllable. Again observing  $[D_1, P_{\mathcal{N}^\perp}] = 0$ , define  $R_\mu : L^2 \rightarrow \mathcal{N}^\perp$  by

$$R_\mu := (D_1 - E_\mu)^{-1}P_{\mathcal{N}^\perp}A . \quad (42)$$

Introducing the operator

$$Q_\mu := 1 - (1 - \mu)R_\mu \quad (43)$$

we get

$$P_{\mathcal{N}}P_{\mathcal{N}_\mu} = Q_\mu P_{\mathcal{N}_\mu} . \quad (44)$$

With the following lemma we shall establish that  $Q_\mu$  is invertible.

**Lemma 6.2** *There exists  $C < \infty$  such that*

$$\|R_\mu\|_{op} < C(1 - \mu)^{-\frac{13}{16}} . \quad (45)$$

**Proof:** Let  $\chi \in L^2$  with  $\|\chi\| = 1$ . Set

$$\xi = P_{\mathcal{N}^\perp}(A\chi) , \quad (46)$$

and note for later that

$$\|\xi\| \leq \|A\|_\infty .$$

Also for later use we define and note that

$$\Phi := P_{\mathcal{N}}(A\chi) , \quad \|\Phi\| \leq \|A\|_\infty . \quad (47)$$

With (42)

$$R_\mu\chi = (D_1 - E_\mu)^{-1}\xi . \quad (48)$$

Set

$$r_\mu = (1 - \mu)^{-\frac{3}{8}} \quad (49)$$

and let  $\mathcal{B}_0(r_\mu) = \{x \in \mathbb{R}^3 : \|x\| < r_\mu\}$ . Choose  $\mu_B < 1$  so that  $\mathcal{S}_A \subset \mathcal{B}_0(r_\mu)$  for all  $\mu \in [\mu_B, 1]$ . For large enough  $r_\mu$  ( $\mu_B$  close enough to one) we have that the vectors  $P_{\mathcal{N}}(\mathbf{1}_{\mathcal{B}(r_\mu)}\Phi^l)$ ,  $1 \leq l \leq n$  are linearly independent, thus  $\{P_{\mathcal{N}}(\mathbf{1}_{\mathcal{B}(r_\mu)}\Phi^l), 1 \leq l \leq n\}$  is a basis of  $\mathcal{N}$ . Hence there exists a  $\tilde{\Phi}_\mu \in \mathcal{N}$  such that

$$P_{\mathcal{N}}(\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu) = P_{\mathcal{N}}(\mathbf{1}_{\mathcal{B}(r_\mu)}\xi) . \quad (50)$$

We define now the spacial cutoff parts

$$\xi_\mu^a := \mathbf{1}_{\mathcal{B}(r_\mu)}\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu \quad (51)$$

$$\xi_\mu^b := \xi - \xi_\mu^a , \quad (52)$$

which are orthogonal to  $\mathcal{N}$  (c.f. (46)). By Schwartz inequality

$$\|\xi_\mu^a\|_1 \leq \|\mathbf{1}_{\mathcal{B}(r_\mu)}\| \|\xi_\mu^a\| \leq \left(\frac{4}{3}\pi r_\mu^3\right)^{\frac{1}{2}} \left(\|\xi\| + \|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\|\right) .$$



Since  $P_{\mathcal{N}}\tilde{\Phi}_\mu = \tilde{\Phi}_\mu$  we have with (50)

$$\begin{aligned}\|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\|^2 &= \langle \mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu, \tilde{\Phi}_\mu \rangle = \langle \mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu, P_{\mathcal{N}}\tilde{\Phi}_\mu \rangle \\ &= \langle P_{\mathcal{N}}\left(\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\right), \tilde{\Phi}_\mu \rangle = \langle \mathbf{1}_{\mathcal{B}(r_\mu)}\xi, \tilde{\Phi}_\mu \rangle \\ &= \langle \mathbf{1}_{\mathcal{B}(r_\mu)}\xi, \mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu \rangle \leq \|\mathbf{1}_{\mathcal{B}(r_\mu)}\xi\| \|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\| ,\end{aligned}$$

hence

$$\|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\| \leq \|\mathbf{1}_{\mathcal{B}(r_\mu)}\xi\| \leq \|\xi\| \leq \|A\|_\infty .$$

Hence there exists a constant  $C < \infty$  so that

$$\|\xi_\mu^a\|_1 \leq \left(\frac{4}{3}\pi\right)^{1/2} 2\|A\|_\infty r_\mu^{3/2} . \quad (53)$$

For (48) we obtain

$$R_\mu\chi = (D_1 - E_\mu)^{-1} \xi_\mu^a + (D_1 - E_\mu)^{-1} \xi_\mu^b$$

and we wish to show that for some  $C < \infty$

$$\|(D_1 - E_\mu)^{-1} \xi_\mu^{a,b}\| < C (1 - \mu)^{-\frac{13}{16}} . \quad (54)$$

“(54) **a**”: We introduce  $\Xi_1(\mathbf{k}, j, \cdot) \in \mathcal{N}$  from Theorem 4.2 (c.f. (15)). Since  $\langle \xi_\mu^a, \Xi_1(\mathbf{k}, j, \cdot) \rangle = 0$  we have

$$\begin{aligned}\mathcal{F}_1(\xi_\mu^a)(\mathbf{k}, j) &:= (2\pi)^{-3/2} \langle \varphi_1(\mathbf{k}, j, \mathbf{x}), \xi_\mu^a \rangle \\ &= (2\pi)^{-3/2} \langle \varphi_1(\mathbf{k}, j, \mathbf{x}) - \Xi_1(\mathbf{k}, j, \mathbf{x}), \xi_\mu^a \rangle .\end{aligned}$$

Thus

$$|\mathcal{F}_1(\xi_\mu^a)(\mathbf{k}, j)| \leq (2\pi)^{-3/2} \|\varphi_1(\mathbf{k}, j, \cdot) - \Xi_1(\mathbf{k}, j, \cdot)\|_\infty \|\xi_\mu^a\|_1 .$$

By using Theorem 4.2 and (53) we get

$$|\mathcal{F}_1(\xi_\mu^a)(\mathbf{k}, j)| \leq C r_\mu^{3/2} . \quad (55)$$

Note that  $D_1\varphi_1(\mathbf{k}, j, \cdot) = E_k\varphi_1(\mathbf{k}, j, \cdot) = \sqrt{k^2 + 1}\varphi_1(\mathbf{k}, j, \cdot)$ . Observing (13) we get, reading in the following  $E_k$  as the symbol for the function of  $\mathbf{k}$  which multiplies the generalized Fourier transform  $\mathcal{F}_1(\xi_\mu^a)(\mathbf{k}, j)$

$$\|(D_1 - E_\mu)^{-1} \xi_\mu^a\| = \left\| \frac{1}{E_k - E_\mu} \mathcal{F}_1(\xi_\mu^a) \right\|$$

Using  $E_k \geq 1 > E_\mu$  and hence  $E_k - E_\mu \geq 1 - E_\mu > 0$ ,  $E_k - E_\mu > E_k - 1 \geq 0$  we compute

$$\begin{aligned}
\| (D_1 - E_\mu)^{-1} \xi_\mu^a \| &\leq \left\| \frac{1}{E_k - E_\mu} \mathbb{1}_{|k| < (1 - E_\mu)^{1/2}} \mathcal{F}_1(\xi_\mu^a) \right\| \\
&\quad + \left\| \frac{1}{E_k - E_\mu} \mathbb{1}_{1 > |k| > (1 - E_\mu)^{1/2}} \mathcal{F}_1(\xi_\mu^a) \right\| \\
&\quad + \left\| \frac{1}{E_k - E_\mu} \mathbb{1}_{|k| > 1} \mathcal{F}_1(\xi_\mu^a) \right\| \\
&\leq \frac{1}{1 - E_\mu} \left\| \mathbb{1}_{|k| < (1 - E_\mu)^{1/2}} \mathcal{F}_1(\xi_\mu^a) \right\| \\
&\quad + \left\| \mathbb{1}_{1 > |k| > (1 - E_\mu)^{1/2}} \frac{1}{E_k - 1} \mathcal{F}_1(\xi_\mu^a) \right\| \\
&\quad + \left\| \mathbb{1}_{|k| > 1} \frac{1}{E_k - 1} \mathcal{F}_1(\xi_\mu^a) \right\|.
\end{aligned}$$

By (55) we obtain with appropriate constants  $C_1, C_2, C_3$

$$\begin{aligned}
\| (D_1 - E_\mu)^{-1} \xi_\mu^a \| &\leq \frac{C_1}{1 - E_\mu} r_\mu^{3/2} (1 - E_\mu)^{3/4} \\
&\quad + C_2 r_\mu^{3/2} \left( \int \mathbb{1}_{1 > |k| > (1 - E_\mu)^{1/2}} (E_k - 1)^{-2} d^3 k \right)^{1/2} \\
&\quad + C_3.
\end{aligned}$$

Noting that  $E_k - 1 \geq k^2/2$  for  $|k| < 1$ , we obtain

$$\begin{aligned}
&\leq r_\mu^{3/2} \left( C_1 (1 - E_\mu)^{-1/4} + C_2 \left( 4\pi \int_{(1 - E_\mu)^{1/2}}^1 4k^{-2} dk \right)^{1/2} \right) + C_3 \\
&= r_\mu^{3/2} \left( C_1 (1 - E_\mu)^{-1/4} + \frac{8\pi}{\sqrt{3}} C_2 \left( (1 - E_\mu)^{-1/2} - 1 \right)^{1/2} \right) + C_3.
\end{aligned}$$

Because  $E_\mu \rightarrow 1$  for  $\mu \rightarrow 1$  and  $r_\mu$  is given by (49) there exists an appropriate constant  $C < \infty$  such that

$$\| (D_1 - E_\mu)^{-1} \xi_\mu^a \| \leq C r_\mu^{3/2} (1 - E_\mu)^{-1/4}. \quad (56)$$

By (36)  $1 - E_\mu \geq \underline{C}(1 - \mu)$ . This and (49) yield (54) a.

“(54) **b**”: By (51) and (52)

$$\|\xi_\mu^b\| = \|\xi - \xi_\mu^a\| \leq \|\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi\| + \|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\|.$$

Since  $\tilde{\Phi}_\mu \in \mathcal{N}$  we have with (50)

$$\begin{aligned} |\langle \mathbf{1}_{\mathcal{B}(r_\mu)}\xi, \tilde{\Phi}_\mu \rangle| &= |\langle \mathbf{1}_{\mathcal{B}(r_\mu)}\xi, P_{\mathcal{N}}\tilde{\Phi}_\mu \rangle| = |\langle P_{\mathcal{N}}\mathbf{1}_{\mathcal{B}(r_\mu)}\xi, \tilde{\Phi}_\mu \rangle| \\ &= |\langle \mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu, \tilde{\Phi}_\mu \rangle| = \|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\|^2. \end{aligned}$$

On the other hand recalling that  $\xi \perp \mathcal{N}$  we obtain by Schwartz inequality

$$|\langle \mathbf{1}_{\mathcal{B}(r_\mu)}\xi, \tilde{\Phi}_\mu \rangle| = |\langle \xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi, \tilde{\Phi}_\mu \rangle| \leq \|\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi\| \|\tilde{\Phi}_\mu\|,$$

hence

$$\|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\| \leq C\|\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi\| \frac{\|\tilde{\Phi}_\mu\|}{\|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\|}.$$

Clearly

$$\lim_{\mu \rightarrow 1} \frac{\|\tilde{\Phi}_\mu\|}{\|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\|} = 1,$$

thus for  $\mu_B$  close enough to 1 there exists a  $C < \infty$  so that

$$\begin{aligned} \|\xi_\mu^b\| &\leq \|\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi\| + \|\mathbf{1}_{\mathcal{B}(r_\mu)}\tilde{\Phi}_\mu\| \\ &\leq C\|\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi\|. \end{aligned} \tag{57}$$

By (46) and the fact that  $A$  has compact support and by (47) with (46) we have for  $r_\mu$  large enough that  $\xi$  is outside the ball  $\mathcal{B}(r_\mu)$  a multiple of  $\Phi$ . Hence  $\xi - \mathbf{1}_{\mathcal{B}(r_\mu)}\xi$  is outside the ball  $\mathcal{B}(r_\mu)$  a multiple of  $\Phi$ . Since  $\Phi \in \mathcal{N}$  with norm bounded in (47) its decay properties are known from the Greens function of  $D_1 - 1$  (see e.g. [15], [22]), namely  $|\Phi| \leq Cx^{-2}$ , we obtain

$$\|\xi_\mu^b\| \leq C \left( \int_{x>r_\mu} x^{-4} d^3x \right)^{\frac{1}{2}} \leq Cr_\mu^{-\frac{1}{2}}$$

with appropriate constants  $C < \infty$ . It follows, again using  $E_k - E_\mu \geq 1 - E_\mu$ , that

$$\begin{aligned} \|(D_1 - E_\mu)^{-1} \xi_\mu^b\| &= \left\| \frac{1}{E_k - E_\mu} \mathcal{F}_1(\xi_\mu^b) \right\| \leq \frac{1}{1 - E_\mu} \|\xi_\mu^b\| \\ &\leq |1 - E_\mu|^{-1} Cr_\mu^{-\frac{1}{2}}. \end{aligned} \tag{58}$$

As before, (36) and (49) yield (54) b.

□

**Corollary 6.3** For  $\mu_B$  close enough to 1

$$\|(1 - \mu)R_\mu\|_{op} \leq (1 - \mu)^{\frac{3}{16}}, \quad \forall \mu \in [\mu_B, 1). \quad (59)$$

Furthermore

$$\lim_{\mu \rightarrow 1} \|Q_\mu\|_{op} = 1$$

and the inverse of the operator (43)

$$Q_\mu^{-1} = \sum_{j=0}^{\infty} (1 - \mu)^j R_\mu^j \quad (60)$$

exists as bounded operator on  $L^2$ .

**Proof:** The Corollary follows immediately from (43) and Lemma 6.2.  $\square$

With this we establish now Lemma 6.1. Let  $n_\mu := \dim \mathcal{N}_\mu$ . Since  $Q_\mu$  is invertible for  $\mu_B$  close enough to 1 it follows by (44) that  $n \geq n_\mu$ .

It is left to show that for  $\mu_B$  close enough to 1  $n \leq n_\mu$  for all  $\mu \in [\mu_B, 1)$ . Assume that for any  $0 < \mu_B < 1$  there exists a  $\mu \in [\mu_B, 1)$  such that  $n > n_\mu$ . Then the  $n_\mu$ -dimensional vectors  $\{v_j, j = 1, \dots, n\}$  defined by their coordinates  $v_j^l := \langle \Phi_\mu^l, \Phi^j \rangle$  are linearly dependent, i.e. there exist nontrivial  $\alpha_j$  such that  $\sum_{j=1}^n \alpha_j v_j = 0$ . Then  $\Phi := \sum_{j=1}^n \alpha_j \Phi^j \neq 0$ ,  $\Phi \in \mathcal{N}$  and  $\Phi$  satisfies  $P_{\mathcal{N}_\mu} \Phi = 0$ . But for any  $\Delta E \in \mathbb{R}$  we have

$$\begin{aligned} \|(D_\mu - 1 + \Delta E)\Phi\|^2 &= \|(D_0 + (\mu - 1)A - 1 + \Delta E)\Phi\|^2 \\ &= \langle (\Delta E - (1 - \mu)A)^2 \Phi, \Phi \rangle \\ &= (\Delta E)^2 + \langle (-2\Delta E(1 - \mu)A + (1 - \mu)^2 A^2) \Phi, \Phi \rangle. \end{aligned}$$

Choosing  $\Delta E = (1 - \mu)\|A\|_\infty < 1$  for  $\mu_B$  close enough to 1, it follows that

$$\|(D_\mu - 1 + \Delta E)\Phi\|^2 < (\Delta E)^2. \quad (61)$$

On the other hand  $P_{\mathcal{N}_\mu} \Phi = 0$  implies by virtue of Condition 2.2 (ii) that  $\Phi$  lies in the absolutely continuous spectrum. Writing  $P_\mu^+$  and  $P_\mu^-$  for the spectral projections onto the positive and negative absolutely continuous spectral subspaces of  $D_\mu$  and using that  $\Delta E < 1$  we obtain

$$\begin{aligned} \|(D_\mu - 1 + \Delta E)\Phi\|^2 &= \|(D_\mu - 1 + \Delta E)P_\mu^+ \Phi\|^2 + \|(D_\mu - 1 + \Delta E)P_\mu^- \Phi\|^2 \\ &\geq \|(1 - 1 + \Delta E)P_\mu^+ \Phi\|^2 + \|(-1 - 1 + \Delta E)P_\mu^- \Phi\|^2 \\ &\geq (\Delta E)^2 \|P_\mu^+ \Phi\|^2 + \|P_\mu^- \Phi\|^2 \\ &\geq (\Delta E)^2 \|P_\mu^+ \Phi\|^2 + (\Delta E)^2 \|P_\mu^- \Phi\|^2 = (\Delta E)^2. \end{aligned}$$

This contradicts to (61) and hence  $n = n_\mu$ .

□

**Corollary 6.4**  $Q_\mu$  is a bijection from  $\mathcal{N}_\mu$  to  $\mathcal{N}$  for  $\mu_B$  close enough to 1.

**Proof:** By (44)  $Q_\mu$  maps  $\mathcal{N}_\mu$  into  $\mathcal{N}$ , thus “onto” by Lemma 6.1 and Corollary 6.3. □

## 6.1 $\mu$ -Derivative of the Bound States

Using  $Q_\mu$  we can construct now for each element  $\Phi \in \mathcal{N}$  a sequence of elements  $(\zeta_\mu)_\mu$  by  $\zeta_\mu := Q_\mu^{-1}\Phi \in \mathcal{N}_\mu$  which is “good” in several respects:

**Corollary 6.5** For  $\mu_B$  close enough to 1 holds: For each  $\Phi \in \mathcal{N}$ ,  $\|\Phi\| = 1$ , the corresponding sequence  $(\zeta_\mu)_{\mu \in [\mu_B, 1]} := (Q_\mu^{-1}\Phi)_{\mu \in [\mu_B, 1]}$  satisfies

(i)  $\zeta_\mu \in \mathcal{N}_\mu$  for all  $\mu \in [1, \mu_B]$ .

(ii)  $P_{\mathcal{N}}\zeta_\mu = \Phi$

(iii)  $\lim_{\mu \rightarrow 0} \|\zeta_\mu\| = 1$ .

We call the sequence  $(\zeta_\mu)_{\mu \in [\mu_B, 1]}$  the good sequence corresponding to  $\Phi$ .

**Proof:** (i) follows from Corollary 6.4.

(ii) follows from (60) using that  $R_\mu$  maps into  $\mathcal{N}^\perp$

(iii) is immediate using (59) in (60).

□

Below we shall need the  $\mu$ -Derivative of  $\Phi_\mu$  given by

**Lemma 6.6** ( $\mu$ -Derivative of the Bound States) Let  $\Phi \in \mathcal{N}$  be normalized and let  $\zeta_\mu$ ,  $\mu \in [\mu_B, 1]$  be the good sequence corresponding to  $\Phi$ . Then<sup>2</sup>

$$\|\partial_\mu \zeta_\mu\| \leq C(1 - \mu)^{-\frac{13}{16}}, \quad \mu \in [\mu_B, 1] \quad (62)$$

with some constant  $C < \infty$ . Since  $\mathcal{N}$  is finite dimensional the constant can be chosen uniformly on  $\mathcal{N}$ .

---

<sup>2</sup>This estimate is not optimal, but sufficient for what is needed later. It seems reasonable to conjecture that the correct exponent is  $-\frac{1}{2}$ .

**Proof:** We shall use the formula (60) to differentiate  $\zeta_\mu = Q_\mu^{-1}\Phi$  (c.f. (60)). Formally

$$\begin{aligned}
\partial_\mu \zeta_\mu &= \partial_\mu \sum_{j=0}^{\infty} ((\mu-1)R_\mu)^j \Phi \\
&= (\partial_\mu(\mu-1)R_\mu) \sum_{j=1}^{\infty} j ((\mu-1)R_\mu)^{j-1} \Phi \\
&= R_\mu \sum_{j=1}^{\infty} j ((\mu-1)R_\mu)^{j-1} \Phi \\
&\quad + (\mu-1)(\partial_\mu R_\mu) \sum_{j=1}^{\infty} j ((\mu-1)R_\mu)^{j-1} \Phi .
\end{aligned}$$

Hence by Corollary 6.3 we obtain rigorously

$$\|\partial_\mu \zeta_\mu\| \leq C (\|R_\mu\|_{op} + \|(\mu-1)\partial_\mu R_\mu\|_{op}) . \quad (63)$$

For the second term observe that formally

$$\begin{aligned}
\partial_\mu R_\mu \chi &= \partial_\mu (D_1 - E_\mu)^{-1} (A\chi - \langle A\chi, \Phi \rangle \Phi) \\
&= (\partial_\mu E_\mu) (D_1 - E_\mu)^{-2} (A\chi - \langle A\chi, \Phi \rangle \Phi) \\
&= (\partial_\mu E_\mu) (D_1 - E_\mu)^{-1} R_\mu \chi ,
\end{aligned}$$

which can be justified using for example the spectral decomposition of  $D_1$ . From this we get

$$\|\partial_\mu R_\mu\|_{op} \leq C(1-\mu)^{-1} \|R_\mu\|_{op}$$

hence

$$\|\partial_\mu \zeta_\mu\| \leq C \|R_\mu\|_{op} \quad (64)$$

and (62) follows with Lemma 6.2.

□

## 7 Proof of Theorem 2.4

We now study the true time evolution  $U^\varepsilon(0, s)$  as given by (5). To prove Theorem 2.4 we have to control the time propagation of  $\psi_s^\varepsilon$ . This propagation is naturally qualitatively different for  $s < 0$  (“adiabatic bound state evolution”) and  $s > 0$  (“scattering”). Hence we control the propagation for  $s < 0$  and  $s > 0$  separately.

## 7.1 Control of $\psi_s^\varepsilon$ for $s \leq 0$

Usually adiabatic theory assumes a spectral gap. Here we are in a situation where the eigenvalue  $E_{\mu(s)}$  will close the gap to the upper continuum. We need to control the adiabatic change of the bound states without a gap condition.

**Lemma 7.1** (*Adiabatic Lemma without a gap*)

Let  $s < 0$  be such that a normalized bound state  $\tilde{\Phi}_{\mu(s)}$  of  $D_{\mu(s)}$  with energy  $E_{\mu(s)} > -1$  exists. Then

$$\lim_{\varepsilon \rightarrow 0} \|P_{\mathcal{N}} U^\varepsilon(0, s) \tilde{\Phi}_{\mu(s)}\| = 1. \quad (65)$$

**Proof:** Let  $s < 0$ ,  $\tilde{\Phi}_{\mu(s)}$  be a normalized bound state. By the adiabatic Theorem [39] we have that for any  $s < s_0 < 0$

$$\lim_{\varepsilon \rightarrow 0} \|P_{\mathcal{N}_{\mu(s_0)}} U^\varepsilon(s_0, s) \tilde{\Phi}_{\mu(s)}\| = 1. \quad (66)$$

Setting  $\tilde{\Phi}_{\mu(s_0)}^\varepsilon := P_{\mathcal{N}_{\mu(s_0)}} U^\varepsilon(s_0, s) \tilde{\Phi}_{\mu(s)}$  we note that (66) is equivalent to

$$\lim_{\varepsilon \rightarrow 0} \|U^\varepsilon(s_0, s) \tilde{\Phi}_{\mu(s)} - \tilde{\Phi}_{\mu(s_0)}^\varepsilon\| = 0. \quad (67)$$

We now define an approximate time evolution

$$\Phi_{s_0, \mu(s)}^\varepsilon := \exp\left(-\frac{i}{\varepsilon} \int_{s_0}^s E_{\mu(v)} dv\right) Q_{\mu(s)}^{-1} Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)} \quad (68)$$

and note that

$$\Phi_{s_0, \mu(s_0)}^\varepsilon = \tilde{\Phi}_{\mu(s_0)}.$$

Now  $\Phi_{s_0} := Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)} / \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\|$  is normalized and in  $\mathcal{N}$  and hence  $\zeta_{s_0, \mu(s)} := Q_{\mu(s)}^{-1} \Phi_{s_0}$  is the good sequence corresponding to  $\Phi_{s_0}$ . (68) becomes

$$\Phi_{s_0, \mu(s)}^\varepsilon = \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\| \exp\left(-\frac{i}{\varepsilon} \int_{s_0}^s E_{\mu(v)} dv\right) \zeta_{s_0, \mu(s)}.$$

We compare the approximate time evolution with the true one

$$\begin{aligned}
\Phi_{s_0, \mu(0)}^\varepsilon - U^\varepsilon(0, s_0) \tilde{\Phi}_{\mu(s_0)} &= \int_{s_0}^0 \partial_u (U^\varepsilon(0, u) \Phi_{s_0, \mu(u)}^\varepsilon) du \\
&= i \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\| \int_{s_0}^0 U^\varepsilon(0, u) \left( \frac{D_u}{\varepsilon} - i \partial_u \right) \exp \left( -\frac{i}{\varepsilon} \int_{s_0}^u E_{\mu(v)} dv \right) \zeta_{s_0, \mu(u)} du \\
&= \frac{i}{\varepsilon} \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\| \int_{s_0}^0 U^\varepsilon(0, u) (D_u - E_{\mu(u)}) \exp \left( -\frac{i}{\varepsilon} \int_{s_0}^u E_{\mu(v)} dv \right) \zeta_{s_0, \mu(u)} du \\
&\quad + \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\| \int_{s_0}^0 U^\varepsilon(0, u) \exp \left( -\frac{i}{\varepsilon} \int_{s_0}^u E_{\mu(v)} dv \right) \partial_u \zeta_{s_0, \mu(u)} du .
\end{aligned}$$

Since  $(D_u - E_{\mu(u)}) \Phi_{\mu(u)} = 0$

$$\begin{aligned}
\Phi_{s_0, \mu(0)}^\varepsilon - U^\varepsilon(0, s_0) \tilde{\Phi}_{\mu(s_0)} &= \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\| \int_{s_0}^0 U^\varepsilon(0, u) \exp \left( -\frac{i}{\varepsilon} \int_{s_0}^u E_{\mu(v)} dv \right) \partial_u \zeta_{s_0, \mu(u)} du .
\end{aligned}$$

Hence by unitarity of  $U^\varepsilon$ , Lemma 6.6 and Condition 2.2 (ii)

$$\begin{aligned}
\|\Phi_{s_0, \mu(0)}^\varepsilon - U^\varepsilon(0, s_0) \tilde{\Phi}_{\mu(s_0)}\| &\leq \|Q_{\mu(s_0)} \tilde{\Phi}_{\mu(s_0)}\| C \int_{s_0}^0 u^{-\frac{13}{16}} du \\
&\leq \|Q_{\mu(s_0)}\|_{op} C \frac{16}{3} s_0^{\frac{3}{16}} \leq C s_0^{\frac{3}{16}}
\end{aligned}$$

by Corollary 6.3. Thus uniformly in  $\varepsilon$

$$\lim_{s_0 \rightarrow 0} \|\Phi_{s_0, \mu(0)}^\varepsilon - U^\varepsilon(0, s_0) \tilde{\Phi}_{\mu(s_0)}\| = 0 . \tag{69}$$

Furthermore we obtain from (69) that uniformly in  $\varepsilon$

$$\lim_{s_0 \rightarrow 0} \left| \|\Phi_{s_0, \mu(0)}^\varepsilon\| - \|U^\varepsilon(0, s_0) \tilde{\Phi}_{\mu(s_0)}\| \right| = \lim_{s_0 \rightarrow 0} \left| \|\Phi_{s_0, \mu(0)}^\varepsilon\| - 1 \right| = 0 ,$$

so that

$$\lim_{s_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \|\Phi_{s_0, \mu(0)}^\varepsilon\| = 1 . \tag{70}$$



Since  $\Phi_{s_0, \mu(0)}^\varepsilon \in \mathcal{N}$

$$\begin{aligned}
\left| \|P_{\mathcal{N}}U^\varepsilon(0, s)\tilde{\Phi}_{\mu(s)}\| - \|\Phi_{s_0, \mu(0)}^\varepsilon\| \right| &\leq \|P_{\mathcal{N}}U^\varepsilon(0, s)\tilde{\Phi}_{\mu(s)} - P_{\mathcal{N}}\Phi_{s_0, \mu(0)}^\varepsilon\| \\
&\leq \|U^\varepsilon(0, s)\tilde{\Phi}_{\mu(s)} - U^\varepsilon(0, s_0)\tilde{\Phi}_{\mu(s_0)}\| \\
&\quad + \|U^\varepsilon(0, s_0)\tilde{\Phi}_{\mu(s_0)} - \Phi_{s_0, \mu(0)}^\varepsilon\| \\
&= \|U^\varepsilon(s_0, s)\tilde{\Phi}_{\mu(s)} - \tilde{\Phi}_{\mu(s_0)}\| \\
&\quad + \|U^\varepsilon(0, s_0)\tilde{\Phi}_{\mu(s_0)} - \Phi_{s_0, \mu(0)}^\varepsilon\|.
\end{aligned}$$

It follows with (67), (69) and (70), that

$$\begin{aligned}
\lim_{\varepsilon \rightarrow 0} \|P_{\mathcal{N}}U^\varepsilon(0, s)\tilde{\Phi}_{\mu(s)}\| &= \lim_{s_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \|P_{\mathcal{N}}U^\varepsilon(0, s)\tilde{\Phi}_{\mu(s)}\| \\
&= \lim_{s_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \|\Phi_{s_0, \mu(0)}^\varepsilon\| = 1.
\end{aligned}$$

□

## 7.2 Propagation Estimates for the time dependent Case: “Short” Times $s > 0$

We shall introduce a time  $\sigma > 0$  which is a time of order one. For example the time at which the switching factor  $\mu(s)$  is half way between 1 and its maximum. Our estimates will be valid until this time. It is in fact the crucial time after which the critical bound state has already left the range of the potential. We shall in the next section consider “long” times, i.e. the times bigger than  $\sigma$ . We shall now consider the auxiliary time evolution (16) on the macroscopic time scale  $s = t\varepsilon$ . That evolution will be denoted by  $V_{\mu(v)}^\varepsilon(s, 0)$  where  $v$  is fixed! It is defined by

$$i\partial_s V_{\mu(v)}^\varepsilon(s, 0) = \frac{1}{\varepsilon} D_{\mu(v)} V_{\mu(v)}^\varepsilon(s, 0). \quad (71)$$

We first reformulate our Corollary 5.2 for  $V_{\mu(v)}^\varepsilon$  be given by (71). Instead of  $\mu \in (1, 1 + \delta]$  we have now  $v \in (0, \sigma]$ . For the chosen  $\sigma$  we can replace in view of (36) the factor  $\mu(v) - 1$  corresponding to  $\mu - 1$  in Corollary 5.2 by  $v$  at little extra costs. We formulate first this adjustment as

**Corollary 7.2** (*Propagation Estimate - stationary case*)

Let  $\mathcal{S} \subset \mathbb{R}^3$  be compact,  $0 < \xi < 1$ . There exists a  $\sigma > 0$  such that there exists a  $C < \infty$  and for all  $m \in \mathbb{N}$  constants  $C_m < \infty$  such that for all  $v \in (0, \sigma]$ , all  $u > \varepsilon(\mu(v) - 1)^{-\frac{3}{2(1-\xi)}}$  and all  $\chi \in L^2$  with  $\text{supp}\chi \subset \mathcal{S}$  the following holds

(i) for  $\underline{\kappa} = \varepsilon^{\frac{1}{2}(1-\xi)}u^{-\frac{1}{2}(1-\xi)}$  and for all  $s \geq u$

$$\|\mathbb{1}_{\mathcal{S}}V_{\mu(v)}^\varepsilon(s, 0)\rho_{\underline{\kappa}, \mu(v)}\chi\| \leq C_m \varepsilon^m s^{-m} (\|D_{\mu(v)}\chi\|),$$

(ii) for  $\underline{\kappa} = \varepsilon^{\frac{1}{2}(1-\xi)}u^{-\frac{1}{2}(1-\xi)}$

$$\|(1 - \rho_{\underline{\kappa}, \mu(v)})\chi\| \leq C \varepsilon^{\frac{3}{4}(1-\xi)}u^{-\frac{3}{4}(1-\xi)}v^{-1/4}\|\chi\|,$$

(iii)

$$\|\mathbb{1}_{\mathcal{S}}V_{\mu(v)}^\varepsilon(u, 0)\chi\| \leq C v^{-1/2} \varepsilon^{\frac{3}{2}(1-\xi)}u^{-\frac{3}{2}(1-\xi)}(\|\chi\| + \|D_{\mu(v)}\chi\|).$$

We shall use that to control the time evolution of a wavefunction under the influence of the time dependent Dirac operator. We wish to note, that in contrast to Corollary 7.2 the estimates in the following Lemmata (i.e. Lemma 7.3 - 7.6 are done for a fixed wave function and not for a whole set of wavefunctions. Hence the constants  $C$  and  $C_m$  below depend in fact on more criteria of  $\chi$  (in particular the energy  $\|D_{\mu(0)}\chi\|$ ) than the  $C$  and  $C_m$  in Corollary 7.2.

**Lemma 7.3** (*Propagation Estimates - time dependent Case: "Short" times*)

Let  $U^\varepsilon(s, u)$  be given by (5). Let  $\mathcal{S} \supset \mathcal{S}_A$  be compact and let  $0 < \xi < 1/3$ . Let  $\chi$  be normalized with  $\text{supp}\chi \subset \mathcal{S}$  and  $\|D_{\mu(0)}\chi\| < \infty$ . Let  $\sigma > 0$  be a possible choice of  $\sigma$  for Corollary 7.2 and such that  $\overline{C} \geq \partial_s \mu(s) \geq \underline{C} > 0$  on  $(0, \sigma]$ . Then there exists  $C < \infty$  such that for all  $s \in (0, \sigma]$

$$\|\mathbb{1}_{\mathcal{S}}U^\varepsilon(s, 0)\chi\| \leq C \left( \varepsilon^{\frac{1}{2}-\frac{3}{2}\xi} s^{-\frac{3}{2}} \right). \quad (72)$$

**Remark 7.4** *This estimate gives the decay time of the critical bound state. It is of the order of  $\varepsilon^{1/3}$ , i.e.  $\varepsilon^{-2/3}$  on the microscopic time scale. One should compare this with the decay of an  $L^2$ -function in a non-critical situation which is of order one on the microscopic time scale.*

**Proof:** Using that  $\chi$  is normalized the Lemma follows trivially for  $s \leq \varepsilon^{1/3-\xi}$  by choosing  $C > 1$ . Let  $s > \varepsilon^{1/3-\xi}$  and

$$\psi_s^\varepsilon := U^\varepsilon(s, 0)\chi. \quad (73)$$

Now  $V_{\mu(s)}^\varepsilon$  is controllable with help of Corollary 7.2. We shall “replace” the propagator  $U^\varepsilon$  by  $V_{\mu(v)}^\varepsilon$ . Then

$$\partial_s U^\varepsilon(s, 0) = \varepsilon^{-1} D_s U^\varepsilon(s, 0) \quad \partial_s V_{\mu(v)}^\varepsilon(s, 0) = \varepsilon^{-1} D_{\mu(v)} V_{\mu(v)}^\varepsilon(s, 0) .$$

and in view of (38)

$$\begin{aligned} U^\varepsilon(s, 0) - V_{\mu(v)}^\varepsilon(s, 0) &= \int_0^s \partial_u (V_{\mu(v)}^\varepsilon(s, u) U^\varepsilon(u, 0)) du \quad (74) \\ &= -\frac{i}{\varepsilon} \int_0^s V_{\mu(v)}^\varepsilon(s, u) (D_{\mu(v)} - D_{\mu(u)}) U^\varepsilon(u, 0) du \\ &= -\frac{i}{\varepsilon} \int_0^s V_{\mu(v)}^\varepsilon(s, u) (\mu(v) - \mu(u)) A U^\varepsilon(u, 0) du . \end{aligned}$$

Hence

$$\begin{aligned} \psi_s^\varepsilon &= U^\varepsilon(s, 0)\chi \\ &= V_{\mu(v)}^\varepsilon(s, 0)\chi + \frac{i}{\varepsilon} \int_0^s (\mu(u) - \mu(v)) V_{\mu(v)}^\varepsilon(s, u) A \psi_u^\varepsilon du . \quad (75) \end{aligned}$$

We shall now choose a “good”  $v$ . The good choice is  $v = s$ . We shall explain why: The “error” coming from  $(\mu(s) - \mu(u)) A \psi_u^\varepsilon$  for  $u$  close to  $s$  is very small. The “error” coming from earlier times is large in  $L^2$ , but the propagation time  $s - u$  is also large and hence most of the wavefunction will have left the region  $\mathcal{S}$  (c.f. Corollary 7.2)). So our strategy is not to show that the “error” is small in  $L^2$  (which would not work) but to show that the “error” which is not small in  $L^2$  leaves the region  $\mathcal{S}$  and what is left of the error in the relevant region is small and thus in fact deserves to be called an error. The estimates in Corollary 7.2 are only valid from a small time on. This is an inheritance of the singular behavior of the generalized eigenfunctions and must be taken into account. This will lead to a slight complication which makes another splitting necessary.

In detail: choosing  $u = s$  we obtain, splitting the time according to the idea above introducing another cutoff  $\tilde{\sigma}$  which will be specified below (and

which takes care of the applicability of the Corollary 7.2)

$$\begin{aligned}\psi_s^\varepsilon &= V_{\mu(s)}^\varepsilon(s, 0)\chi + \frac{i}{\varepsilon} \int_{s-\tilde{\sigma}}^s (\mu(u) - \mu(s)) V_{\mu(s)}^\varepsilon(s, u) A \psi_u^\varepsilon du \\ &\quad + \frac{i}{\varepsilon} \int_0^{s-\tilde{\sigma}} (\mu(u) - \mu(s)) V_{\mu(s)}^\varepsilon(s, u) A \psi_u^\varepsilon du .\end{aligned}$$

Hence

$$\begin{aligned}\|\mathbb{1}_S \psi_s^\varepsilon\| &\leq \|\mathbb{1}_S V_{\mu(s)}^\varepsilon(s, 0)\chi\| + \frac{1}{\varepsilon} \int_{s-\tilde{\sigma}}^s (\mu(s) - \mu(u)) \|A \psi_u^\varepsilon\| du \\ &\quad + \frac{1}{\varepsilon} \int_0^{s-\tilde{\sigma}} (\mu(s) - \mu(u)) \|\mathbb{1}_S V_{\mu(s)}^\varepsilon(s, u) A \psi_u^\varepsilon\| du .\end{aligned}\quad (76)$$

The second summand is bounded by

$$\frac{\bar{C}}{\varepsilon} \int_{s-\tilde{\sigma}}^s (s-u) \|A\|_\infty \|\mathbb{1}_{S_A} \psi_u^\varepsilon\| du \leq \frac{C}{\varepsilon} \tilde{\sigma}^2 .\quad (77)$$

For (76) we wish to apply now Corollary 7.2 (iii) to the first and third term. For the first term

$$\|\mathbb{1}_S V_{\mu(s)}^\varepsilon(s, 0)\chi\|$$

we have the replacement of variables  $u \hat{=} s$  and  $v \hat{=} s$ , hence we need that  $s > \varepsilon(\mu(s) - 1)^{-\frac{3}{2(1-\xi)}} \geq \varepsilon(\bar{C}s)^{-\frac{3}{2(1-\xi)}}$ , i.e. that

$$s^{1+\frac{3}{2(1-\xi)}} > \bar{C}^{-\frac{3}{2(1-\xi)}} \varepsilon = C\varepsilon .\quad (78)$$

But since  $s > \varepsilon^{1/3-\xi}$  we have that  $s^{1+\frac{3}{2(1-\xi)}} > \varepsilon^{(1+\frac{3}{2(1-\xi)})(1/3-\xi)}$ . Since  $\left(1 + \frac{3}{2(1-\xi)}\right)(1/3-\xi) = \frac{5-2\xi}{6} \frac{1-3\xi}{1-\xi} \leq \frac{5}{6} < 1$  the condition for the Corollary is fulfilled provided that  $\varepsilon$  is small enough ( $C\varepsilon^{1/6} < 1$ , with the  $C$  of formula (79)). Hence

$$\|\mathbb{1}_S V_{\mu(s)}^\varepsilon(s, 0)\chi\| \leq C s^{-1/2} \varepsilon^{\frac{3}{2}-\frac{3}{2}\xi} s^{-\frac{3}{2}+\frac{3}{2}\xi} (\|\chi\| + \|D_{\mu(s)}\chi\|) \leq C \varepsilon^{\frac{3}{2}-\frac{3}{2}\xi} s^{-2+\frac{3}{2}\xi} ,\quad (79)$$

since  $\|\chi\| = 1$  and

$$\|D_{\mu(s)}\chi\| \leq |\mu(s) - \mu(0)| \|A\chi\| + \|D_0\chi\| \leq C \|A\|_\infty + \|D_{\mu(0)}\chi\| \leq C .$$

Next we use the Corollary 7.2 to the third term of (76), that is on the term  $\|\mathbf{1}_{\mathcal{S}} V_{\mu(s)}^\varepsilon(s, u) A \psi_u^\varepsilon\|$  with the replacement of variables  $u \hat{=} s - u$  and  $v \hat{=} s$  and  $\chi \hat{=} A \psi_u^\varepsilon$ , which has support  $\mathcal{S}_A \subset \mathcal{S}$ . Therefore we need that

$$s - u > \varepsilon(\mu(s) - 1)^{-\frac{3}{2(1-\varepsilon)}} > C \varepsilon s^{-\frac{3}{2(1-\varepsilon)}}. \quad (80)$$

Choosing

$$\tilde{\sigma} = C \varepsilon s^{-\frac{3}{2(1-\varepsilon)}} \quad (81)$$

with the  $C$  of formula (80), this is satisfied for all  $u < s - \tilde{\sigma}$ , i.e. for the integrand of the third summand. Hence we have for the third term that

$$\|\mathbf{1}_{\mathcal{S}} V_{\mu(s)}^\varepsilon(s, u) A \psi_u^\varepsilon\| \leq s^{-1/2} \varepsilon^{\frac{3}{2} - \frac{3}{2}\xi} (s - u)^{-\frac{3}{2} + \frac{3}{2}\xi} (\|A \psi_u^\varepsilon\| + \|D_{\mu(0)} A \psi_u^\varepsilon\|) \quad (82)$$

To control this term we need to find an upper bound on  $\|D_{\mu(s)} A \psi_u^\varepsilon\|$  uniform in  $0 \leq s, u \leq \sigma$ , which we will do next. We have that (the differential symbol  $\partial_u$  stands also for  $\frac{d}{du}$ )

$$\begin{aligned} & |\partial_s \langle \psi_s^\varepsilon, D_{\mu(s)} \psi_s^\varepsilon \rangle| \\ &= \left| \langle \psi_s^\varepsilon, (\partial_s D_{\mu(s)}) \psi_s^\varepsilon \rangle + \langle (\partial_s \psi_s^\varepsilon), D_{\mu(s)} \psi_s^\varepsilon \rangle + \langle \psi_s^\varepsilon, D_{\mu(s)} \partial_s \psi_s^\varepsilon \rangle \right| \\ &= \left| \langle \psi_s^\varepsilon, A(\partial_s \mu(s)) \psi_s^\varepsilon \rangle + \left\langle \frac{i}{\varepsilon} D_{\mu(s)} \psi_s^\varepsilon, D_{\mu(s)} \psi_s^\varepsilon \right\rangle + \langle \psi_s^\varepsilon, D_{\mu(s)} \frac{i}{\varepsilon} D_{\mu(s)} \psi_s^\varepsilon \rangle \right| \\ &\leq (\partial_s \mu(s)) \|A\|_\infty \|\mathbf{1}_{\mathcal{S}_A} \psi_s^\varepsilon\|. \end{aligned}$$

Integrating and observing that  $\|D_{\mu(0)} \chi\| < \infty$  implies  $|\langle \chi, D_{\mu(0)} \chi \rangle| < \infty$ , we obtain  $|\langle \psi_s^\varepsilon, D_{\mu(s)} \psi_s^\varepsilon \rangle| < C$  with a constant  $C$  uniform in  $0 \leq s \leq \sigma$ . Using this we get similarly that  $|\langle \psi_s^\varepsilon, D_{\mu(s)}^2 \psi_s^\varepsilon \rangle| < C$  with another constant  $C$  uniform in  $0 \leq s \leq \sigma$ . Hence with (38)

$$\begin{aligned} \|D_{\mu(s)} A \psi_u^\varepsilon\| &\leq \|(\mu(s) - \mu(u)) A^2 \psi_u^\varepsilon\| + \|A D_{\mu(u)} \psi_u^\varepsilon\| + \left\| \sum_{j=1}^3 \alpha_j (\partial_j A) \psi_u^\varepsilon \right\| \\ &\leq (\mu(s) - \mu(u)) \|A\|_\infty^2 + \|A\|_\infty \|D_{\mu(u)} \psi_u^\varepsilon\| + \|\nabla A\|_\infty \leq C \quad (83) \end{aligned}$$

with a constant  $C$  uniform in  $0 \leq s \leq \sigma$ . Using (82), (83) and that  $\psi_u^\varepsilon$  is

normalized we get

$$\begin{aligned}
& \frac{1}{\varepsilon} \int_0^{s-\tilde{\sigma}} (\mu(s) - \mu(u)) \|\mathbf{1}_S V_{\mu(s)}^\varepsilon(s, u) A\psi_u^\varepsilon\| du \\
& \leq \frac{C}{\varepsilon} \int_0^{s-\tilde{\sigma}} (\mu(s) - \mu(u)) \\
& \quad s^{-1/2} \varepsilon^{\frac{3}{2}-\frac{3}{2}\xi} (s-u)^{-\frac{3}{2}+\frac{3}{2}\xi} (\|A\psi_u^\varepsilon\| + \|D_{\mu(s)} A\psi_u^\varepsilon\|) du \\
& \leq \frac{\overline{CC}}{\varepsilon} \int_0^s s^{-1/2} \varepsilon^{\frac{3}{2}-\frac{3}{2}\xi} (s-u)^{-\frac{1}{2}+\frac{3}{2}\xi} (\|A\|_\infty + C) du \\
& \leq C \varepsilon^{\frac{1}{2}-\frac{3}{2}\xi} s^{\frac{3}{2}\xi}.
\end{aligned}$$

This, (77) (with  $\tilde{\sigma}$  replaced by (81)) and (79) introduced in (76) yields

$$\begin{aligned}
\|\mathbf{1}_S \psi_s^\varepsilon\| & \leq C \varepsilon^{\frac{3}{2}-\frac{3}{2}\xi} s^{-2+\frac{3}{2}\xi} + C \varepsilon s^{-\frac{3}{1-\xi}} + C \varepsilon^{\frac{1}{2}-\frac{3}{2}\xi} s^{\frac{3}{2}\xi} \\
& \leq C_\xi \varepsilon^{\frac{1}{2}-\frac{3}{2}\xi} s^{-\frac{3}{2}} \left( \varepsilon s^{-\frac{1}{2}+\frac{3}{2}\xi} + \varepsilon^{\frac{1}{2}+\frac{3}{2}\xi} s^{\frac{-3-3\xi}{2-2\xi}} + s^{\frac{3}{2}+\frac{3}{2}\xi} \right).
\end{aligned}$$

Since  $\sigma > s > \varepsilon^{1/3-\xi}$  it follows that for  $\varepsilon$  small enough

$$\begin{aligned}
\varepsilon s^{-\frac{1}{2}+\frac{3}{2}\xi} & \leq \varepsilon \varepsilon^{-\frac{1}{6}} \sigma^{\frac{3}{2}\xi} < 1 \\
\varepsilon^{\frac{1}{2}+\frac{3}{2}\xi} s^{\frac{-3-3\xi}{2-2\xi}} & < \varepsilon^{\frac{1}{2}+\frac{3}{2}\xi} \varepsilon^{-\frac{1+\xi}{2} \frac{1-3\xi}{1-\xi}} \leq \varepsilon^{\frac{1}{2}+\frac{3}{2}\xi} \varepsilon^{-\frac{1+\xi}{2}} = \varepsilon^\xi < 1 \\
s^{\frac{3}{2}+\frac{3}{2}\xi} & < \sigma^{\frac{3}{2}+\frac{3}{2}\xi} < C.
\end{aligned}$$

Hence

$$\|\mathbf{1}_S \psi_s^\varepsilon\| \leq C \varepsilon^{\frac{1}{2}-\frac{3}{2}\xi} s^{-\frac{3}{2}}.$$

□

### 7.3 Propagation Estimates for the Time Dependent Case: “Long” Times $s > 0$

Lemma 7.3 gives estimates on the decay behavior for times smaller than  $\sigma$ . In principle the Lemma can be extended also for larger times for a very large class of potentials  $A_{\mu(s)}$ . This seems alright as long as the propagator  $V_{\mu(s)}^\varepsilon$  leads to fast enough decay, i.e. as long as  $\mu(s)$  is bounded away from one.

But we are especially interested in the case, that  $\mu(s)$  attains the critical value  $\mu(s) = 1$  again after time  $\sigma$ , since the potential will be switched off

again. We shall need a different technique to estimate the decay behavior in this situation for times  $s > \sigma$  (c.f. Lema 7.5). This techniques will be based on the fact that by time  $\sigma$  most of the wavefunction has already left the area  $\mathcal{S}_A$  of the potential. This allows us to chose in the comparison of  $U^\varepsilon(s, 0)\chi$  with  $V_v^\varepsilon\chi$  a fixed value of  $v$ , in fact we shall use  $v = \sigma$ , in contrast to Lemma 7.3 where we chose  $v = s$ . This has the advantage, that we can use fixed cutoffs in Fourier space, i.e. we can use Corollary 7.2 (i) and (ii).

**Lemma 7.5** (*Propagation Estimates - Time Dependent Case: “Long” times*)

Let  $\mathcal{S}$ ,  $\sigma$ ,  $\xi$  and  $\|\chi\|$  satisfy the conditions of Lemma 7.3. Then there exists a  $C < \infty$  such that for all  $s > \sigma$

$$\|\mathbf{1}_{\mathcal{S}}U^\varepsilon(s, 0)\chi\| \leq C\varepsilon^{\frac{1}{12}-\frac{3}{4}\xi}. \quad (84)$$

**Proof:** Despite the fact that an  $L^2$ -function has mostly left any compact region by time  $\sigma$ , to show that it scatters is still not easy. The reason is that we deal with a time evolution which is generated by a time dependent Hamiltonian. We shall use again a freezing of the potential defining an auxiliary time evolution. We start with an auxiliary lemma about the auxiliary time evolution with which we shall later compare the true evolution:

**Lemma 7.6** (*Auxiliary Lemma*)

Let  $\mathcal{S}$ ,  $\sigma$ ,  $\xi$  and  $\|\chi\|$  satisfy the conditions of Lemma 7.3. Let  $\tilde{U}^\varepsilon(s)$  be the unitary defined by  $\tilde{U}^\varepsilon(s, 0) = U^\varepsilon(s, 0)$  for  $s \leq \sigma$  and  $\tilde{U}^\varepsilon(s, \sigma) = V_{\mu(\sigma)}^\varepsilon(s, \sigma)$  for  $s > \sigma$ . Let

$$\chi_s^\varepsilon := \tilde{U}^\varepsilon(s, 0)\chi. \quad (85)$$

Then there exists a  $\tilde{\psi}_s^\varepsilon$ , a  $C < \infty$  and for any  $m \in \mathbb{N}$  a  $C_m$  such that for all  $s > \sigma$

$$\|\chi_s^\varepsilon - \tilde{\psi}_s^\varepsilon\| \leq C\varepsilon^{\frac{1}{12}-\frac{3}{4}\xi} \quad (86)$$

and

$$\|\mathbf{1}_{\mathcal{S}}\tilde{\psi}_s^\varepsilon\| \leq C_m\varepsilon^{m/3-1}s^{-m}. \quad (87)$$

**Proof:** With the notation (73)  $\chi_s^\varepsilon = \psi_s^\varepsilon$  for  $s \leq \sigma$  and  $\chi_s^\varepsilon = V_{\mu(\sigma)}^\varepsilon(s, \sigma)\chi_\sigma^\varepsilon$  for  $s > \sigma$ . Using (75) with  $s = v = \sigma$  we obtain

$$\chi_\sigma^\varepsilon = V_{\mu(\sigma)}^\varepsilon(\sigma, 0)\chi + \frac{i}{\varepsilon} \int_0^\sigma (\mu(v) - \mu(\sigma)) V_{\mu(\sigma)}^\varepsilon(\sigma, v)A(x)\psi_v^\varepsilon dv.$$

Hence applying  $V_{\mu(\sigma)}^\varepsilon(s, \sigma)$  yields

$$\begin{aligned}
\chi_s^\varepsilon &= V_{\mu(\sigma)}^\varepsilon(s, 0)\chi + \frac{i}{\varepsilon} \int_0^\sigma (\mu(v) - \mu(\sigma)) V_{\mu(\sigma)}^\varepsilon(s, v) A(x) \psi_v^\varepsilon dv \\
&= V_{\mu(\sigma)}^\varepsilon(s, 0)\chi + \frac{i}{\varepsilon} \int_0^{\sigma - \varepsilon^{2/3}} (\mu(v) - \mu(\sigma)) V_{\mu(\sigma)}^\varepsilon(s, v) A(x) \psi_v^\varepsilon dv \\
&\quad + \frac{i}{\varepsilon} \int_{\sigma - \varepsilon^{2/3}}^\sigma (\mu(v) - \mu(\sigma)) V_{\mu(\sigma)}^\varepsilon(s, v) A(x) \psi_v^\varepsilon dv . \tag{88}
\end{aligned}$$

The splitting of the integrals are done for application of Corollary 7.2 (i) and (ii) to control (88) and will become clearer in a moment. We must process in various steps. We define (in view of Corollary 7.2) now the function  $\tilde{\psi}_s^\varepsilon$  of lemma 7.6.

$$\begin{aligned}
\tilde{\psi}_s^\varepsilon &:= V_{\mu(\sigma)}^\varepsilon(s, 0) \rho_{\underline{\kappa}, \mu(\sigma)} \chi \\
&\quad + \frac{i}{\varepsilon} \int_0^{\sigma - \varepsilon^{2/3}} (\mu(v) - \mu(\sigma)) V_{\mu(\sigma)}^\varepsilon(s, v) \rho_{\underline{\kappa}, \mu(\sigma)} A \psi_v^\varepsilon dv \tag{89}
\end{aligned}$$

We note that by definition

$$\tilde{\psi}_s^\varepsilon = \tilde{U}^\varepsilon(s, \sigma) \tilde{\psi}_\sigma^\varepsilon . \tag{90}$$

Now

$$\begin{aligned}
\|\mathbf{1}_S \tilde{\psi}_s^\varepsilon\| &\leq \|\mathbf{1}_S V_{\mu(\sigma)}^\varepsilon(s, 0) \rho_{\underline{\kappa}, \mu(\sigma)} \chi\| \\
&\quad + \frac{1}{\varepsilon} \int_0^{\sigma - \varepsilon^{2/3}} (\mu(v) - \mu(\sigma)) \|\mathbf{1}_S V_{\mu(\sigma)}^\varepsilon(s, v) \rho_{\underline{\kappa}, \mu(\sigma)} A \psi_v^\varepsilon\| dv . \tag{91}
\end{aligned}$$

We subtract now (89) from (88), we take the norms, use triangle inequality and use unitarity of  $V_{\mu(\sigma)}^\varepsilon$

$$\begin{aligned}
\|\chi_s^\varepsilon - \tilde{\psi}_s^\varepsilon\| &\leq \|\chi - \rho_{\underline{\kappa}, \mu(\sigma)} \chi\| \\
&\quad + \frac{1}{\varepsilon} \int_0^{\sigma - \varepsilon^{2/3}} |\mu(v) - \mu(\sigma)| \|A \psi_v^\varepsilon - \rho_{\underline{\kappa}, \mu(\sigma)} A \psi_v^\varepsilon\| dv \\
&\quad + C \frac{1}{\varepsilon} \int_{\sigma - \varepsilon^{2/3}}^\sigma |\mu(v) - \mu(\sigma)| \|A \psi_v^\varepsilon\| dv .
\end{aligned}$$



Using that  $\|A\psi_v^\varepsilon\| \leq \|A\|_\infty$  one gets after trivial reordering

$$\begin{aligned} \|\chi_s^\varepsilon - \tilde{\psi}_s^\varepsilon\| &\leq \|(1 - \rho_{\underline{\kappa}, \mu(\sigma)}) \chi\| + C\varepsilon^{1/3} \\ &\quad + \frac{C\overline{C}}{\varepsilon} \int_0^{\sigma - \varepsilon^{2/3}} (\sigma - v) \|(1 - \rho_{\underline{\kappa}, \mu(\sigma)}) A\psi_v^\varepsilon\| dv. \end{aligned} \quad (92)$$

We shall now estimate the terms in (91) and (92) using Corollary 7.2. The terms are  $\|\mathbb{1}_{\mathcal{S}} V_{\mu(\sigma)}^\varepsilon(s, 0) \rho_{\underline{\kappa}, \mu(\sigma)} \chi\|$ ,  $\|(1 - \rho_{\underline{\kappa}, \mu(\sigma)}) \chi\|$ ,  $\|\mathbb{1}_{\mathcal{S}} V_{\mu(\sigma)}^\varepsilon(s, v) \rho_{\underline{\kappa}, \mu(\sigma)} A\psi_v^\varepsilon\|$  and  $\|(1 - \rho_{\underline{\kappa}, \mu(\sigma)}) A\psi_v^\varepsilon\|$ .

Note that  $\chi$  and  $A\psi_v^\varepsilon$  are compactly supported and have finite energy (by (83) and the assumptions of the lemma). For application of the Corollary 7.2 we must check whether the inequality for the propagation time (i.e.  $s \geq u > \varepsilon(\mu(v) - 1)^{-\frac{3}{2(1-\varepsilon)}}$ ) is satisfied.

We first want to use the Corollary 7.2 (i) and (ii) on  $\chi$  with the following replacements of variables:  $s \hat{=} s$ ,  $v \hat{=} \sigma$  and  $u \hat{=} \sigma$ . Hence the condition of the Corollary reads now  $s \geq \sigma > \varepsilon(\mu(\sigma) - 1)^{-\frac{3}{2(1-\varepsilon)}}$ . The first inequality is satisfied by assumption of the lemma. Since  $\partial_v \mu(v) \geq \underline{C} > 0$  for all  $0 < v < \sigma$  (by assumption of the lemma) and  $\mu(0) = 1$  we have that  $\mu(\sigma) - 1 > 0$ . Hence for small enough  $\varepsilon$  we have that  $\sigma > \varepsilon(\mu(\sigma) - 1)^{-\frac{3}{2(1-\varepsilon)}}$  and Corollary 7.2 (i) and (ii) yields, observing the replacements

$$\|\mathbb{1}_{\mathcal{S}} V_{\mu(\sigma)}(s, 0) \rho_{\underline{\kappa}, \mu(\sigma)} \chi\| \leq C_m (\|D_{\mu(\sigma)} \chi\|) \varepsilon^m s^{-m} \quad (93)$$

and

$$\|(1 - \rho_{\underline{\kappa}, \mu(\sigma)}) \chi\| \leq C \sigma^{-\frac{3}{4}(1-\varepsilon)} \varepsilon^{\frac{3}{4}(1-\varepsilon)} \sigma^{-1/4} \|\chi\| = C \sigma^{-1 + \frac{3}{4}\varepsilon} \varepsilon^{\frac{3}{4}(1-\varepsilon)}. \quad (94)$$

Next we want to use Corollary 7.2 (i) and (ii) on  $\|\mathbb{1}_{\mathcal{S}} V_{\mu(\sigma)}(s, v) \rho_{\underline{\kappa}, \mu(\sigma)} A\psi_v^\varepsilon\|$  with  $v \leq \sigma - \varepsilon^{2/3}$ . Thus we must make the following replacements of variables in the corollary:  $\chi \hat{=} A(x) \psi_v^\varepsilon$ ,  $v \hat{=} \sigma$ ,  $u \hat{=} \sigma - v$  and  $s \hat{=} s - v$ . Then the condition of the Corollary becomes  $s - v \geq \sigma - v > \varepsilon(\mu(\sigma) - 1)^{-\frac{3}{2(1-\varepsilon)}}$ , which is why we did the splitting of the integrals in (88) in the first place, namely we have that  $v \leq \sigma - \varepsilon^{2/3}$ , so that the condition is satisfied for small enough  $\varepsilon$ . Hence we can use the Corollary on  $A(x) \psi_v^\varepsilon$  making the correct replacements to obtain

$$\|\mathbb{1}_{\mathcal{S}} V_{\mu(\sigma)}(s, v) \rho_{\underline{\kappa}, \mu(\sigma)} A\psi_v^\varepsilon\| \leq C_m (\|D_{\mu(\sigma)} A(x) \psi_v^\varepsilon\|) (s - v)^{-m} \varepsilon^m \quad (95)$$

and

$$\|(1 - \rho_{\underline{\kappa}, \mu(\sigma)})A\psi_v^\varepsilon\| \leq C\varepsilon^{\frac{3}{4}(1-\xi)}(\sigma - v)^{-\frac{3}{4}(1-\xi)}\sigma^{-\frac{1}{4}}\|A\psi_v^\varepsilon\|. \quad (96)$$

(93)-(96) can now be used to control (91) and (92). Inserting (94) and (96) into (92) yields

$$\begin{aligned} \|\chi_s^\varepsilon - \tilde{\psi}_s^\varepsilon\| &\leq C\sigma^{-1+\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)} + C\varepsilon^{1/3} \\ &\quad + \frac{C}{\varepsilon} \int_0^{\sigma-\varepsilon^{2/3}} (\sigma - v)(\sigma - v)^{-\frac{3}{4}(1-\xi)}\sigma^{-\frac{1}{4}}\varepsilon^{\frac{3}{4}(1-\xi)}\|A\psi_v^\varepsilon\|dv. \end{aligned}$$

Now comes Lemma 7.3 into play. Without the control of  $\|A\psi_v^\varepsilon\|$  which the lemma provides us with, the last summand would be of order  $\varepsilon^{-1/4}$  and thus explodes as  $\varepsilon \rightarrow 0$ . But the estimates of Lemma 7.3 are only good for times larger than  $\varepsilon^{1/3}$ . For smaller times the trivial estimate  $\|A\psi_v^\varepsilon\| \leq C$  is better. Thus we split the  $v$  integral accordingly and arrive at

$$\begin{aligned} \|\chi_s^\varepsilon - \tilde{\psi}_s^\varepsilon\| &\leq C\sigma^{-1+\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)} + C\varepsilon^{1/3} \\ &\quad + \frac{C}{\varepsilon} \int_0^{\varepsilon^{1/3}} (\sigma - v)^{\frac{1}{4}+\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)}\sigma^{-1/4}\|A\psi_v^\varepsilon\|dv \\ &\quad + \frac{C}{\varepsilon} \int_{\varepsilon^{1/3}}^\sigma (\sigma - v)^{\frac{1}{4}+\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)}\sigma^{-1/4}\|A\psi_v^\varepsilon\|dv. \end{aligned}$$

Now use Lemma 7.3 on the last summand. Observe first that by (73)

$$\|A\psi_v^\varepsilon\| \leq \|A\|_\infty \|\mathbf{1}_{\mathcal{S}_A}U^\varepsilon(v, 0)\chi\| \leq \|A\|_\infty \|\mathbf{1}_{\mathcal{S}_A \cup \mathcal{S}}U^\varepsilon(v, 0)\chi\|$$

We apply to  $\|\mathbf{1}_{\mathcal{S}_A \cup \mathcal{S}}U^\varepsilon(v, 0)\chi\|$  Lemma 7.3 and get, estimating  $\sigma - v \leq \sigma \leq C$ , that for all  $s \geq \sigma$

$$\begin{aligned} \|\chi_s^\varepsilon - \tilde{\psi}_s^\varepsilon\| &\leq C\sigma^{-1+\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)} + C\varepsilon^{1/3} + \frac{C}{\varepsilon}\sigma^{\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)}\varepsilon^{1/3}dv \\ &\quad + \frac{C}{\varepsilon} \int_{\varepsilon^{1/3}}^\sigma (\sigma - v)^{\frac{1}{4}+\frac{3}{4}\xi}\varepsilon^{\frac{3}{4}(1-\xi)}\sigma^{-1/4} \left( \varepsilon^{\frac{1}{2}-\frac{3}{2}\xi}v^{-\frac{3}{2}} \right) dv \\ &\leq C\varepsilon^{\frac{3}{4}-\frac{3}{4}\xi} + C\varepsilon^{1/3} + C\varepsilon^{\frac{1}{12}-\frac{3}{4}\xi} + C\varepsilon^{\frac{1}{12}-\frac{9}{4}\xi} \\ &\leq C\varepsilon^{\frac{1}{12}-\frac{3}{4}\xi}, \end{aligned}$$

which is (86).

Next we estimate (91). Introducing (93) and (95) yields

$$\begin{aligned} \|\mathbf{1}_S \tilde{\psi}_s^\varepsilon\| &\leq C_m (\|D_{\mu(\sigma)} \chi\|) \varepsilon^m s^{-m} \\ &\quad + \frac{\overline{C}}{\varepsilon} \int_0^{\sigma - \varepsilon^{2/3}} (\sigma - v) C_m (\|D_{\mu(\sigma)} A \psi_v^\varepsilon\|) \varepsilon^m (s - v)^{-m} dv. \end{aligned} \quad (97)$$

Recall that  $s \geq \sigma$ , so for  $\varepsilon$  small enough we have

$$s(1 - \varepsilon^{2/3} \sigma^{-1}) \geq \sigma(1 - \varepsilon^{2/3} \sigma^{-1}),$$

hence

$$s - \sigma + \varepsilon^{2/3} \geq \varepsilon^{2/3} s \sigma^{-1},$$

so for  $v \leq \sigma - \varepsilon^{2/3}$

$$(s - v)^{-m} \leq (s - \sigma + \varepsilon^{2/3})^{-m} \leq \varepsilon^{-\frac{2}{3}m} s^{-m} \sigma^m.$$

Using this and the fact that  $\|D_{\mu(\sigma)} A \psi_v^\varepsilon\|$  is bounded (c.f. (83)) we get for (97)

$$\|\mathbf{1}_S \tilde{\psi}_s^\varepsilon\| \leq C_m \varepsilon^m s^{-m} + C_m \varepsilon^{m/3-1} s^{-m} \leq 2C_m \varepsilon^{m/3-1} s^{-m},$$

which is (87).

□

We shall now prove Lemma 7.5. Using (74) we have for  $s > \sigma$  that

$$\left( U^\varepsilon(s, \sigma) - \tilde{U}(s, \sigma) \right) \tilde{\psi}_\sigma^\varepsilon = -\frac{i}{\varepsilon} \int_\sigma^s U^\varepsilon(s, v) (\mu(\sigma) - \mu(v)) A(x) \tilde{U}^\varepsilon(v, \sigma) \tilde{\psi}_\sigma^\varepsilon dv$$

and therefore by (90)

$$\|\mathbf{1}_S U^\varepsilon(s, \sigma) \tilde{\psi}_\sigma^\varepsilon\| \leq \|\mathbf{1}_S \tilde{\psi}_\sigma^\varepsilon\| + \frac{1}{\varepsilon} \int_\sigma^s (\mu(v) - \mu(\sigma)) \|A\|_\infty \|\mathbf{1}_S \tilde{\psi}_v^\varepsilon\| dv.$$

Using (87)

$$\begin{aligned} \|\mathbf{1}_S U^\varepsilon(s, \sigma) \tilde{\psi}_\sigma^\varepsilon\| &\leq C_m \varepsilon^{m/3-1} s^{-m} + \frac{C_m}{\varepsilon} \int_\sigma^s (\mu(\sigma) - \mu(v)) v^{-m} \varepsilon^{m/3-1} dv \\ &\leq C_m \varepsilon^{m/3-1} s^{-m} + \frac{C_m}{\varepsilon} \int_\sigma^s v^{-m+1} \varepsilon^{m/3-1} dv \\ &\leq C_m \varepsilon^{m/3-1} s^{-m} + C_m (s^{-m+2} - \sigma^{-m+2}) \varepsilon^{m/3-2} \\ &\leq C_m \varepsilon^{m/3-1} s^{-m} + C_m s^{-m+2} \varepsilon^{m/3-1-1} \leq C_m s^{-m} \varepsilon^{m/3-2} \end{aligned} \quad (98)$$

We turn now to  $\|\mathbf{1}_S U^\varepsilon(s, 0)\chi\|$ . Recall that  $\chi_\sigma^\varepsilon = \tilde{U}^\varepsilon(\sigma, 0)\chi = U^\varepsilon(\sigma, 0)\chi$ , thus

$$\begin{aligned} \|\mathbf{1}_S U^\varepsilon(s, 0)\chi\| &= \|\mathbf{1}_S U^\varepsilon(s, \sigma)\chi_\sigma^\varepsilon\| \\ &\leq \left\| \mathbf{1}_S U^\varepsilon(s, \sigma) \left( \tilde{\psi}_\sigma^\varepsilon - \chi_\sigma^\varepsilon \right) \right\| + \left\| \mathbf{1}_S U^\varepsilon(s, \sigma) \tilde{\psi}_\sigma^\varepsilon \right\| \\ &\leq \left\| \tilde{\psi}_\sigma^\varepsilon - \chi_\sigma^\varepsilon \right\| + \left\| \mathbf{1}_S U^\varepsilon(s, \sigma) \tilde{\psi}_\sigma^\varepsilon \right\| \\ &\leq C\varepsilon^{\frac{1}{12} - \frac{3}{4}\xi} + C_m s^{-m} \varepsilon^{m/3-2}, \end{aligned}$$

where we used (86) and (98). Choosing  $m$  large enough the Lemma follows.

## 7.4 Proof of APC

We come now to the proof of Theorem 2.4. We wish to establish that for  $s > 0$  and  $\chi \in L^2$  :  $\lim_{\varepsilon \rightarrow 0} \langle \psi_s^\varepsilon, \chi \rangle = 0$ . From Lemma 7.1 we have that  $\lim_{\varepsilon \rightarrow 0} \|(1 - P_{\mathcal{N}})\psi_0^\varepsilon\| = 0$ . Therefore by

$$\lim_{\varepsilon \rightarrow 0} \langle \psi_s^\varepsilon, \chi \rangle = \lim_{\varepsilon \rightarrow 0} \langle U^\varepsilon(s, 0)P_{\mathcal{N}}\psi_0^\varepsilon, \chi \rangle + \lim_{\varepsilon \rightarrow 0} \langle U^\varepsilon(s, 0)(1 - P_{\mathcal{N}})\psi_0^\varepsilon, \chi \rangle,$$

and

$$\lim_{\varepsilon \rightarrow 0} \langle U^\varepsilon(s, 0)(1 - P_{\mathcal{N}})\psi_0^\varepsilon, \chi \rangle \leq \lim_{\varepsilon \rightarrow 0} \|(1 - P_{\mathcal{N}})\psi_0^\varepsilon\| = 0$$

Theorem 2.4 follows from

**Corollary 7.7** (*Decay of the Critical Bound State*)

Let  $s > 0$  and  $\chi \in L^2$ . Then

$$\lim_{\varepsilon \rightarrow 0} |\langle U^\varepsilon(s, 0)P_{\mathcal{N}}\psi_0^\varepsilon, \chi \rangle| = 0. \quad (99)$$

**Proof:** Note that  $P_{\mathcal{N}}$  projects on the subspace with energy 1, hence  $\|D_0 P_{\mathcal{N}}\psi_0^\varepsilon\| \leq 1$ . For the proof it is very convenient to use a two scale argument. Let  $j(\mathbf{x}) \in C^\infty$  be a mollifier with  $j(\mathbf{x}) = 1$  for  $x \leq 1$  and  $j(\mathbf{x}) = 0$  for  $x \geq 2$ , define for any  $\delta > 0$   $j_\delta(\mathbf{x}) := j(\delta\mathbf{x})$ ,  $\chi_{\delta,\varepsilon}^1 := j_\delta P_{\mathcal{N}}\psi_0^\varepsilon$  and  $\chi_\delta^2 := j_\delta \chi$ . Note that this definition yields, that

$$\lim_{\delta \rightarrow 0} \|P_{\mathcal{N}}\psi_0^\varepsilon - \chi_{\delta,\varepsilon}^1\| = 0 = \lim_{\delta \rightarrow 0} \|\chi - \chi_\delta^2\| \quad (100)$$

and

$$\begin{aligned} \|D_0 \chi_{\delta,\varepsilon}^1\| &= \|D_0 j_\delta P_{\mathcal{N}}\psi_0^\varepsilon\| \leq C \sup_{k=1,2,3} \|\partial_k j_\delta\|_\infty \|P_{\mathcal{N}}\psi_0^\varepsilon\| + \|j_\delta D_0 P_{\mathcal{N}}\psi_0^\varepsilon\| \\ &= C\delta + \|j_\delta P_{\mathcal{N}}\psi_0^\varepsilon\| \leq C\delta + \|\psi_0^\varepsilon\| < \infty. \end{aligned}$$

We shall now use Lemma 7.3 and Lemma 7.5. Let  $s > 0$  and  $\delta > 0$ . Set  $\xi = 1/12$  and let  $\mathcal{S}_\delta$  be the support of  $j_\delta$ . Let  $\{\Phi_l\}$  be an orthonormal basis of  $\mathcal{N}$ . For each  $\Phi_l$  there exists a  $C_{\delta,l} < \infty$  such that

$$\|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) j_\delta \Phi_l\| \leq C_{\delta,l} \varepsilon^{\frac{1}{48}}.$$

Let  $C_\delta := \max_{l=1,\dots,n} \{C_{\delta,l}\}$ , then the operator  $\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) j_\delta P_{\mathcal{N}}$  is a matrix on  $\mathcal{N}$  with norm bounded by

$$\|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) j_\delta P_{\mathcal{N}}\|_{op} \leq n C_\delta \varepsilon^{\frac{1}{48}}.$$

Thus

$$\|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) \chi_{\delta,\varepsilon}^1\| = \|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) j_\delta P_{\mathcal{N}} \psi_0^\varepsilon\| \leq n C_\delta \varepsilon^{\frac{1}{48}} \|P_{\mathcal{N}} \psi_0^\varepsilon\| \leq n C_\delta \varepsilon^{\frac{1}{48}}.$$

Hence

$$\begin{aligned} |\langle U^\varepsilon(s, 0) P_{\mathcal{N}} \psi_0^\varepsilon, \chi \rangle| &\leq |\langle U^\varepsilon(s, 0) P_{\mathcal{N}} \psi_0^\varepsilon, \chi_\delta^2 \rangle| + |\langle U^\varepsilon(s, 0) P_{\mathcal{N}} \psi_0^\varepsilon, \chi - \chi_\delta^2 \rangle| \\ &\leq |\langle j_\delta U^\varepsilon(s, 0) P_{\mathcal{N}} \psi_0^\varepsilon, \chi \rangle| + \|P_{\mathcal{N}} \psi_0^\varepsilon\| \|\chi - \chi_\delta^2\| \\ &\leq \langle \mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) P_{\mathcal{N}} \psi_0^\varepsilon, \chi \rangle + \|\chi - \chi_\delta^2\| \\ &\leq \|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) P_{\mathcal{N}} \psi_0^\varepsilon\| \|\chi\| + \|\chi - \chi_\delta^2\| \\ &\leq \|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) \chi_{\delta,\varepsilon}^1\| + \|\mathbf{1}_{\mathcal{S}_\delta} U^\varepsilon(s, 0) (P_{\mathcal{N}} \psi_0^\varepsilon - \chi_{\delta,\varepsilon}^1)\| \\ &\quad + \|\chi - \chi_\delta^2\| \\ &= C_\delta \varepsilon^{\frac{1}{48}} + \|P_{\mathcal{N}} \psi_0^\varepsilon - \chi_{\delta,\varepsilon}^1\| + \|\chi - \chi_\delta^2\|. \end{aligned}$$

Taking first the limit  $\varepsilon \rightarrow 0$  and then  $\delta \rightarrow 0$  the Corollary follows in view of (100).  $\square$

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## 8 Appendix: Proof of Lemma 5.1 (23)

Recall (31)

$$\begin{aligned} &V_\mu(t, 0) \rho_{\underline{k}, \mu} (1 - \rho_{\bar{k}, \mu}) \chi(\mathbf{x}) \\ &= \sum_{j=1}^4 \int (2\pi)^{-\frac{3}{2}} \exp(-itE_k) \varphi_\mu(\mathbf{k}, j, \mathbf{x}) \widehat{\rho}_{\underline{k}} (1 - \widehat{\rho}_{\bar{k}}) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) d^3k. \end{aligned}$$

We estimate the right hand side via stationary phase method, i.e. we integrate by parts. Using  $\frac{iE_k}{kt} \partial_k \exp(-itE_k) = \exp(-itE_k)$  in partial integrations yield - writing

$$\left(\partial_k \frac{E_k}{k}\right)^m := \partial_k \frac{E_k}{k} \partial_k \frac{E_k}{k} \dots ,$$

where  $\partial_k$  acts on everything to the right -

$$\begin{aligned} & V_\mu(t, 0) \rho_{\underline{\kappa}, \mu} (1 - \rho_{\bar{\kappa}, \mu}) \chi(\mathbf{x}) \\ &= \left(-\frac{i}{t}\right)^m \sum_{j=1}^4 \int_0^\infty \int (2\pi)^{-\frac{3}{2}} \exp(-itE_k) \\ & \quad \left( \left(\partial_k \frac{E_k}{k}\right)^m \varphi_\mu(\mathbf{k}, j, \mathbf{x}) \widehat{\rho}_{\underline{\kappa}} (1 - \widehat{\rho}_{\bar{\kappa}}) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) k^2 \right) d\Omega dk \\ &= \left(-\frac{i}{t}\right)^m \sum_{j=1}^4 \int k^{-2} (2\pi)^{-\frac{3}{2}} \exp(-itE_k) \\ & \quad \left( \left(\partial_k \frac{E_k}{k}\right)^m \varphi_\mu(\mathbf{k}, j, \mathbf{x}) \widehat{\rho}_{\underline{\kappa}} (1 - \widehat{\rho}_{\bar{\kappa}}) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) k^2 \right) d^3 k \end{aligned}$$

Since  $\widehat{\rho}_{\underline{\kappa}}(\mathbf{k}) = 0$  for  $k \leq \underline{\kappa}$  and  $k \geq \bar{\kappa}$

$$\begin{aligned} \|\mathbf{1}_S V_\mu(u, 0) \chi\|_\infty &\leq t^{-m} \frac{4}{3} \pi \bar{\kappa}^3 4 \tag{101} \\ & \sup_{2\bar{\kappa} > k > \underline{\kappa}, \mathbf{x} \in S, j} \left| k^{-2} \left( \left(\partial_k \frac{E_k}{k}\right)^m \varphi_\mu(\mathbf{k}, j, \mathbf{x}) \widehat{\rho}_{\underline{\kappa}} (1 - \widehat{\rho}_{\bar{\kappa}}) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) k^2 \right) \right|_\infty . \end{aligned}$$

We next show that for any  $j, l, r \in \mathbb{N}_0$  there exist  $C_{j,l,r}$  so that

$$\left(\partial_k \frac{E_k}{k}\right)^m k^2 f(k) = \sum_{j+l+r=m} C_{j,l,r} E_k^{m-2r} k^{-m-l+r+2} \partial_k^j f(k) . \tag{102}$$

We prove this equation by induction over  $m$ . For  $m = 0$  (102) follows trivially. Assume that (102) holds for some  $m \in \mathbb{N}$ . It follows that

$$\begin{aligned}
\left(\partial_k \frac{E_k}{k}\right)^{m+1} k^2 f(k) &= \partial_k \frac{E_k}{k} \left(\partial_k \frac{E_k}{k}\right)^m k^2 f(k) \\
&= \partial_k \frac{E_k}{k} \sum_{j+l+r=m} C_{j,l,r} E_k^{m-2r} k^{-m+2-l+r} \partial_k^j f(k) \\
&= \partial_k \sum_{j+l+r=m} C_{j,l,r} E_k^{m-2r+1} k^{(-m-1+2)-l+r} \partial_k^j f(k) \\
&= \sum_{j+l+r=m} C_{j,l,r} (\partial_k E_k^{m-2r+1}) k^{(-m-1+2)-l+r} \partial_k^j f(k) \\
&\quad + \sum_{j+l+r=m} C_{j,l,r} E_k^{m-2r+1} (\partial_k k^{(-m-1+2)-l+r}) \partial_k^j f(k) \\
&\quad + \sum_{j+l+r=m} C_{j,l,r} E_k^{m-2r+1} k^{-m+3-l+r} \partial_k^{j+1} f(k)
\end{aligned}$$

Using that  $E_k = \sqrt{k^2 + 1}$  we have that

$$\partial_k E_k^m = m E_k^{m-1} \partial_k \sqrt{k^2 + 1} = m E_k^{m-2} k.$$

Setting  $\tilde{m} = m + 1$ ,  $\tilde{j} = j + 1$ ,  $\tilde{l} = l + 1$  and  $\tilde{r} = r + 1$  yields

$$\begin{aligned}
\left(\partial_k \frac{E_k}{k}\right)^{m+1} k^2 f(k) &= \sum_{j+l+\tilde{r}=\tilde{m}} C_{j,l,\tilde{r}} E_k^{\tilde{m}-2\tilde{r}} k^{-\tilde{m}+2-l+\tilde{r}} \partial_k^j f(k) \\
&\quad + \sum_{j+\tilde{l}+\tilde{r}=\tilde{m}} C_{j,\tilde{l},\tilde{r}} E_k^{\tilde{m}-2\tilde{r}} k^{-\tilde{m}+2-\tilde{l}+\tilde{r}} \partial_k^j f(k) \\
&\quad + \sum_{\tilde{j}+\tilde{l}+\tilde{r}=\tilde{m}} C_{\tilde{j},\tilde{l},\tilde{r}} E_k^{\tilde{m}-2\tilde{r}} k^{-\tilde{m}+2-l+\tilde{r}} \partial_k^{j+1} f(k)
\end{aligned}$$

for appropriate  $C_{\tilde{j},l,\tilde{r}} < \infty$ ,  $C_{j,\tilde{l},\tilde{r}} < \infty$  and  $C_{j,l,\tilde{r}} < \infty$ , and (102) follows for  $\tilde{m} = m + 1$ . Induction yields that (102) holds for all  $m \in \mathbb{N}_0$ .

Note that for  $k \rightarrow 0$   $k^{-2} E_k^{m-2r} k^{-m+2-l+r}$  is of order  $k^{-m-l+r}$ . For  $k \rightarrow \infty$   $E_k$  is of order  $k$ , hence  $k^{-2} E_k^{m-2r} k^{-m+2-l+r}$  is of order  $k^{-l-r}$  (hence bounded for large  $k$ ). Since we only observe  $\underline{k} \rightarrow 0$  it follows with (102) that for any  $m, j \in \mathbb{N}_0$  there exist  $C_{m,j} < \infty$  such that

$$|k^2 \left( \partial_k \frac{E_k}{k} \right)^m k^2 f(k)| \leq \sum_{j=0}^m C_{m,j} k^{-2m+j} |\partial_k^j f(k)|. \quad (103)$$

In our case (c.f. (101)) we have  $f = \varphi_\mu \widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}}) \mathcal{F}_\mu(\chi)$ . Using the product rule of differentiation it follows that

$$\begin{aligned} & \partial_k^j \varphi_\mu \widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}}) \mathcal{F}_\mu(\chi) \\ &= \sum_{j_1+j_2+j_3+j_4=j} C_{j_1, j_2, j_3, j_4} (\partial_k^{j_1} \varphi_\mu) (\partial_k^{j_2} \widehat{\rho}_{\underline{\kappa}}) (1 - \partial_k^{j_3} \widehat{\rho}_{\overline{\kappa}}) (\partial_k^{j_4} \mathcal{F}_\mu(\chi)), \end{aligned}$$

where  $C_{j_1, j_2, j_3, j_4}$  is a combinatorial factor. With (19), (32) and (33) we get using that  $\underline{\kappa} < \overline{\kappa}$

$$\begin{aligned} |\partial_k^j \varphi_\mu \widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}}) \mathcal{F}_\mu(\chi)| &< \sum_{j_1+j_2+j_3+j_4=j} C_{j_1, j_2, j_3, j_4} C_{j_1} C_{j_4} \underline{\kappa}^{-j_2-j_3} (1+x)^{j_1} \\ &\quad \left( \underline{\kappa}^{-1} + \left| \sum_{l=1}^n \frac{k}{|\mu-1-\nu_l k^2| + ck^3} \right| \right)^{j_1+j_4+2}. \end{aligned}$$

Collecting the worst terms (i.e. handling the two cases  $\underline{\kappa}^{-1} < \left| \sum_{l=1}^n \frac{k}{|\mu-1-\nu_l k^2| + ck^3} \right|$  and “ $\geq$ ” separately) we get with an appropriate constant  $C_j$  that

$$\begin{aligned} & |\partial_k^j \varphi_\mu \widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}}) \mathcal{F}_\mu(\chi)| \\ &< C_j (1+x)^j \left( \underline{\kappa}^{-j-2} + \left| \sum_{l=1}^n \frac{k}{|\mu-1-\nu_l k^2| + ck^3} \right|^{j+2} \right). \end{aligned}$$

With (103), again collecting the worst terms, it follows that

$$\begin{aligned} |k^2 \left( \partial_k \frac{E_k}{k} \right)^m k^2 \varphi_\mu \widehat{\rho}_{\underline{\kappa}}(1 - \widehat{\rho}_{\overline{\kappa}}) \mathcal{F}_\mu(\chi)| &< (1+x)^m C_m (\underline{\kappa}^{-m-2} k^{-m} + k^{-2m}) \\ &\quad + (1+x)^m C_m k^2 \left| \sum_{l=1}^n \frac{1}{|\mu-1-\nu_l k^2| + ck^3} \right|^{m+2}. \end{aligned}$$



Thus (recall that  $\underline{\kappa} < 1$  and that  $\mathcal{S}$  is compactly supported, hence  $(1+x)^m$  is bounded by some constant)

$$\begin{aligned} & \sup_{2\bar{\kappa} > k > \underline{\kappa}, \mathbf{x} \in \mathcal{S}, j} \left| k^{-2} \left( \left( \partial_k \frac{E_k}{k} \right)^m \varphi_\mu(\mathbf{k}, j, \mathbf{x}) \widehat{\rho}_{\underline{\kappa}} (1 - \widehat{\rho}_{\bar{\kappa}}) \mathcal{F}_\mu(\chi)(\mathbf{k}, j) k^2 \right) \right|_\infty \quad (104) \\ & < C_m \left( \underline{\kappa}^{-2m-2} + \sup_{2\bar{\kappa} > k > \underline{\kappa}} \left( k^2 \left| \sum_{l=1}^n \frac{1}{|\mu - 1 - \nu_l k^2| + ck^3} \right|^{m+2} \right) \right). \end{aligned}$$

Since  $\underline{\kappa} < 1$  and thus

$$\sup_{2\bar{\kappa} > k > \underline{\kappa}} \left\{ k^2 \left| \sum_{l=1}^n \frac{1}{|\mu - 1 - \nu_l k^2| + ck^3} \right|^2 \right\} < \frac{1}{c^2 \underline{\kappa}^4},$$

(104) is bounded from above by

$$C_m \underline{\kappa}^{-4} \left( \underline{\kappa}^{-2m} + \sup_{2\bar{\kappa} > k > \underline{\kappa}} \left| \sum_{l=1}^n \frac{1}{|\mu - 1 - \nu_l k^2| + ck^3} \right|^m \right).$$

With (101) (and using that for positive  $a, b$  and  $m \in \mathbb{N}$  we have  $(a+b)^m \geq a^m + b^m$ ) equation (23) follows.

□

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