Section 2.2 Subobject classifier

Motivation If D: Set, then $\mathcal{P}(D) \cong 2^D$

$\chi : \mathcal{P}(D) \to 2^D$

$A \mapsto \chi_A$

$\chi_A : D \to 2 \qquad \chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{ow} \end{cases}$, completely determ. $(x \in A \vee x \notin A)$

- χ is 1-1 : $\chi_A = \chi_B \to A = B$ let $x \in A \to \chi_A(x) = 1 = \chi_B(x) \to x \in B$, similarly $B \subseteq A$.
- χ is onto 2^D: let $f \in 2^D$ and $A_f = \{x \in D \mid f(x) = 1\}$. Then $\chi_{A_f} = f$, since

  $\chi_{A_f}(x) = 1 \leftrightarrow x \in A_f \leftrightarrow f(x) = 1$.

Hence $A_f = f^{-1}(\{1\})$, and by Examples 1.8.2 the square

$$\begin{array}{ccc} A_f & \xrightarrow{I_{A_f}} & \{1\} \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ D & \xrightarrow[f]{} & 2 \end{array}$$

is pullback. We can rewrite the diagram

> If $f : A \to B$, $C \subseteq B$
>
> $$\begin{array}{ccc} f^{-1}(C) & \xrightarrow{f\,!} & C \\ {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle i} \\ A & \xrightarrow[f]{} & B \end{array}$$
>
> $A \times_B C \cong f^{-1}(C)$

$$\begin{array}{ccc} A_f & \xrightarrow{i'} & D \\ {\scriptstyle I_{A_f}}\downarrow & & \downarrow{\scriptstyle f} \\ \{1\} & \xrightarrow[i]{} & 2 \end{array} \quad \text{and} \quad \begin{array}{ccc} A_f & \xrightarrow{i'} & D \\ {\scriptstyle I_{A_f}}\downarrow & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[\text{true}]{} & 2 \end{array}$$

(just interchange the sides of the initial corner)

$1 = \{0\}$

$\text{true}(0) = 1$

and the last square is again a pullback: let $f \circ g = \text{true} \circ I_B$, hence

$f(g(b)) = 1, \ \forall b \in B \Rightarrow$

$g(b) \in A_f, \ \forall b \in B$

i.e., the function $g : B \to D$ provided by the data is actually $B \to A_f$.

We define $k = g$ and the diagram commutes, uniqueness is immediate.

$$\begin{array}{ccc} B & \xrightarrow{g} & D \\ {\scriptstyle k}\searrow & & \\ & A_f \xrightarrow{i'} & D \\ {\scriptstyle I_B} & \downarrow{\scriptstyle I_{A_f}} & \downarrow{\scriptstyle f} \\ & 1 \xrightarrow[\text{true}]{} & 2 \end{array}$$

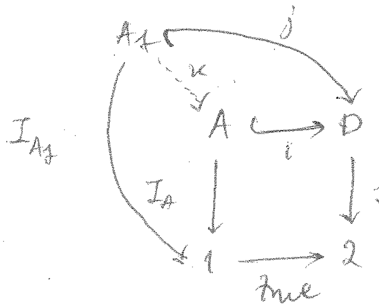
If ~~hence~~ f ~~is arbitrary~~ $= \chi_A$, then $A_{\chi_A} = A$ ~~, we also get that~~

and the above pullback becomes

$$\begin{array}{ccc} A & \hookrightarrow & D \\ {\scriptstyle I_A}\downarrow & & \downarrow{\scriptstyle \chi_A} \\ 1 & \xrightarrow[\text{true}]{} & 2 \end{array}$$

Remark 2.2.1 (Characterization of χ_A). χ_A is the unique arrow-function $D \to 2$ that makes the above square a pullback.

Proof: Let $f: D \to 2$ s.t. the square becomes a pullback. Take as competitor-cone A_f with the obvious maps j and I_{A_f}. Clearly

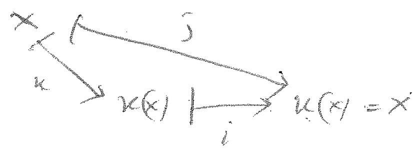


if $x \in A_f$, $(f \circ j)(x) = f(x) = 1 = \text{true}(I_{A_f}(x))$

The unique $x: A_f \to A$ has the property

$i \circ x = j$ (and clearly $I_A \circ x = I_{A_f}$)

But i, j are inclusions, i.e., if $x \in A_f$



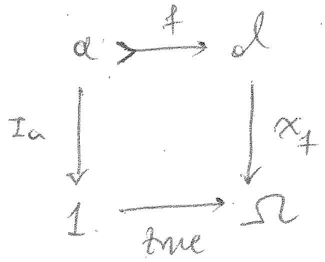
i.e., x is an inclusion, hence $A_f \subseteq A$.

Of course, if $y \in A$, then $f(i(y)) = f(y) = \text{true}(I_A(y)) = 1$. i.e., $y \in A_f$. Hence $A \subseteq A_f$ and eventually $A = A_f$. Since $f = \chi_{A_f}$ we conclude $f = \chi_A$. □

Hence, the set 2 together with the function $\text{true}: 1 \to 2$ provide (through the notion of pullback) a characterization of characteristic functions in Set.

Definition 2.2.2 Let $\mathcal{C}$ a cat with 1.
A <u>subobject classifier</u> for $\mathcal{C}$ is ~~(Ω, true) s.t.~~ Ω: obc together with
$\text{true}: 1 \to \Omega$ s.t. the following is satisfied

<u>Ω-axiom</u>. For every monic $f: a \rightarrowtail d$, there is <u>unique</u> $\chi_f: d \to \Omega$ s.t. tfdps (the following diagram is a pullback square)

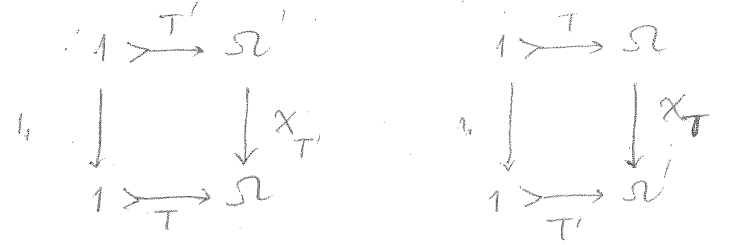


The arrow χ_f is the <u>characteristic arrow</u>, or the <u>character</u> of the pseudo-subobject f of d.

We also use the notation $\text{true} \equiv T$

• **Proposition 2.2.3** A subobject classifier is unique up to isomorphism. (monic on the domain 1) We have $\Omega \cong \Omega'$.

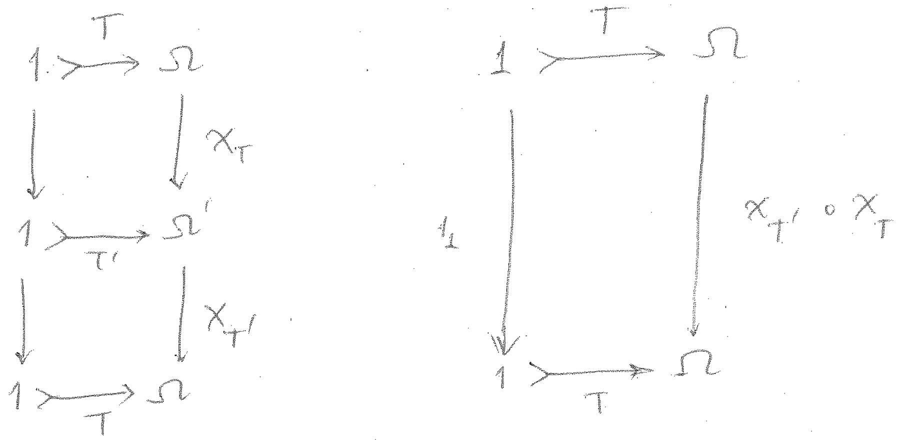
Proof: Let $(\Omega, T : 1 \to \Omega)$ and $(\Omega', T' : 1 \to \Omega')$ subobject classifiers. Then



hence by PBL $\Longrightarrow$

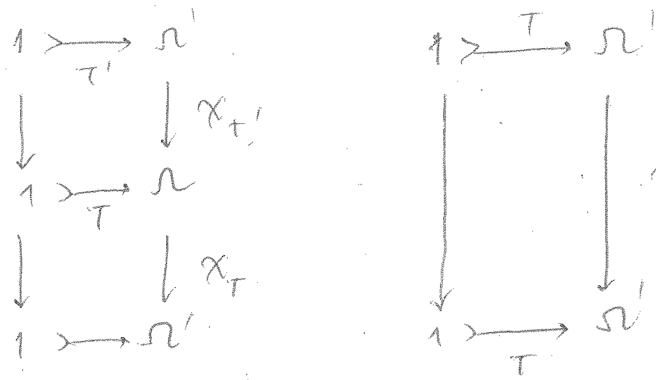
(Ω, T) is subobject class. $\Rightarrow$ pullback $(T = \chi_{T'} \circ T')$

(Ω', T') is subobject classifier $\Rightarrow$ pullback $(T' = \chi_T \circ T)$



$\chi_{T'} \circ \chi_T$ is pullback

By the uniqueness of χ_f in Ω-axiom, $\chi_{T'} \circ \chi_T = 1_\Omega$. Similarly, using



$\chi_T \circ \chi_{T'}$, we get $\chi_T \circ \chi_{T'} = 1_{\Omega'}$

i.e. $\underline{\Omega \cong \Omega'}$.

Corollary 2.2.4 $T' = \chi_T \circ T$ or $T = \chi_{T'} \circ T'$, where χ_T is an iso between their codomains.

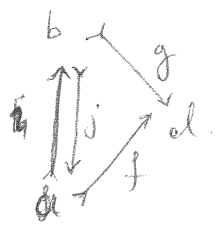
i.e. the classifiers are isomorphic and the true-arrows of the pairs are obtained from each other by composing with an iso between their codomains.

Theorem 22.5 If $f : a \rightarrowtail d$, $g : b \rightarrowtail d$, then

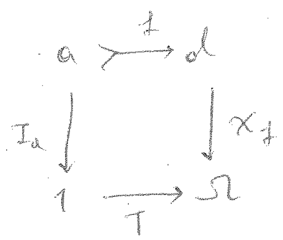
$$f \simeq g \Leftrightarrow \chi_f = \chi_g$$

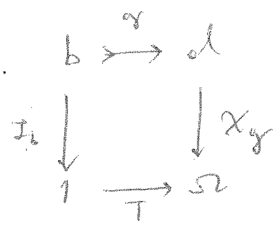
Proof: ($\Leftarrow$) Let $\chi_f = \chi_g$. We show $f \subseteq g$ and $g \subseteq f$ i.e., there are i, j s.t.

(and of course then $b \cong a$)

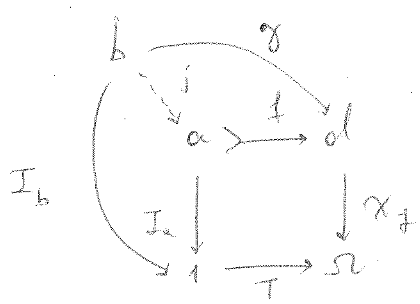


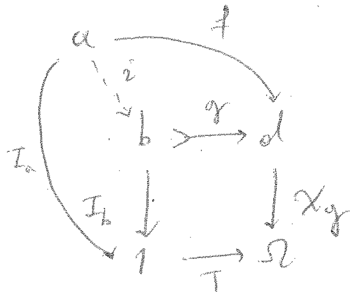
By the Ω-axiom holds





Since want to find an arrow $b \to a$ and $a \to b$ we can do so by using the univ. property of a pullback square





$\chi_f \circ g = \chi_g \circ g = T \circ I_b$

Hence, there is $j : b \to a$ s.t.

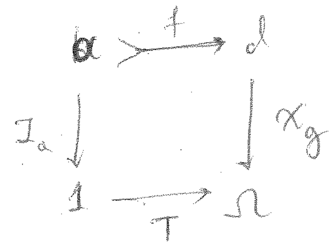
$f \circ j = g$

$\chi_g \circ f = \chi_f \circ f = T \circ I_a$

There is $i : a \to b$ s.t.

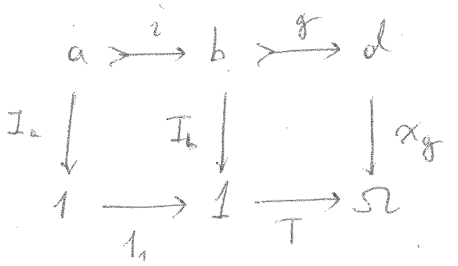
$g \circ i = f$

($\Rightarrow$) Let f, g connected through i, j as in the top first diagram above. By uniqueness of χ_f in Ω-axiom we need to show



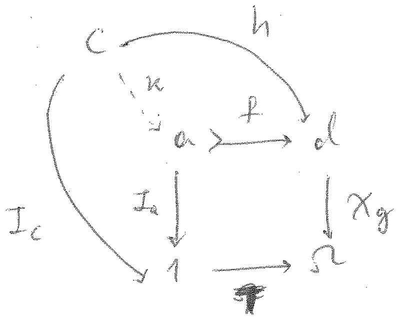
is a pullback square, hence $\chi_g = \chi_f$.

First we need to show that $\chi_g \circ f = T \circ I_a$. We rewrite the previous diagram as follows:



Then we have
$$\begin{aligned}\chi_g \circ f &= \chi_g \circ (g \circ i)\\ &= (\chi_g \circ g) \circ i\\ &= (T \circ I_b) \circ i\\ &= T \circ (I_b \circ i)\\ &= T \circ I_a\end{aligned}$$

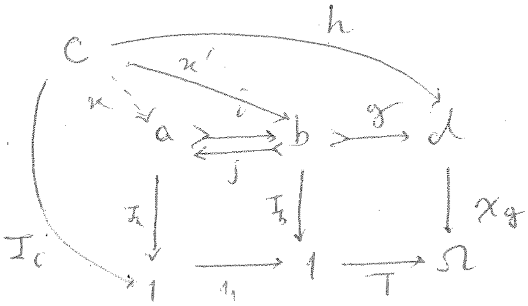
since $a \overset{i}{\rightarrowtail} b \xrightarrow{I_b} 1$ $I_b \circ i$ has to be the unique arrow I_a. Next we show that the required square has the universal pullback property. Let



s.t. $\chi_g \circ h = T \circ I_c$

We find $x: c \to a$ which makes the diag commute

We rewrite things as follows



Since $\chi_g \circ h = T \circ I_c$ we get
$$\chi_g \circ h = T \circ (1_1 \circ I_c)$$
Since the right square is pullback, there is unique $x': c \to b$ s.t.
$$g \circ x' = h \quad \text{and} \quad I_b \circ x' = 1_1 \circ I_c = I_c$$

We define $x = j \circ x'$. Then

$$\begin{aligned}f \circ x &= (g \circ i) \circ j \circ x'\\ &= g \circ (i \circ j) \circ x'\\ &= g \circ x'\\ &= h\end{aligned}$$

$$\begin{aligned}I_a \circ x &= (I_b \circ i) \circ x\\ &= (I_b \circ i) \circ (j \circ x')\\ &= I_b \circ x'\\ &= I_c\end{aligned}$$

□

Proposition 2.2.6 In a cat $\mathcal{C}$ with 1, pullback, Ω, if d is obj, then

$$Sub_{\mathcal{C}}(d) \cong Arr_{\mathcal{C}}(d, \Omega)$$

(in analogy to $Sub(D) \cong \wp(D) \cong 2^D$)

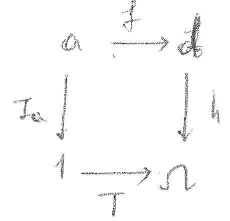
Proof: $e : Sub(d) \longrightarrow Arr(d, \Omega)$

$[f] \mapsto \chi_f$

e is well-defined by Theorem 2.2.5.

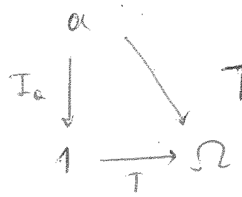
e is 1-1: $[f] = [g] \Leftrightarrow f \simeq g \rightarrow \chi_f = \chi_g \Leftrightarrow e([f]) = e([g])$

e is onto $Arr_{\mathcal{C}}(d, \Omega)$: Let $h : Arr(d, \Omega)$ i.e., $h : d \to \Omega$. Since $\mathcal{C}$ has pullback let the pullback square corresponding to $1 \xrightarrow{T} \Omega \xleftarrow{h} d$



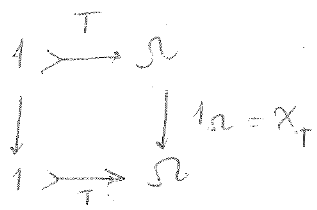
$T : 1 \to \Omega$ is monic. Moreover, by Prp. 1.8.7. f is also monic. By Ω-axiom (and the uniqueness there) $h = \chi_f$, and $e([f]) = h$. □

Notation

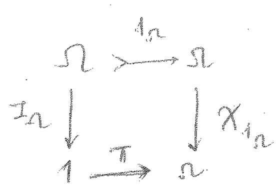


$T \circ I_a = T_a$ (or true$_a$)

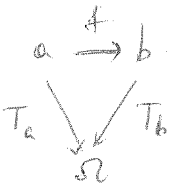
Ex. (i) $\chi_T = 1_\Omega$



(ii) $\chi_{1_\Omega} = T \circ I_\Omega = T_\Omega$



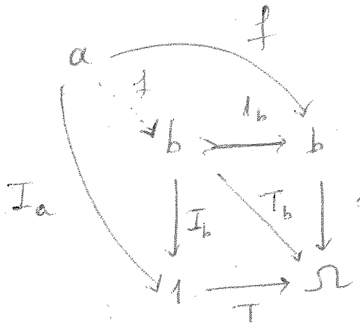
(iii) If $f : a \to b$, then



is commutative i.e., $T_b \circ f = T_a$

Proposition 2.2.7

Proof of (iii)



NO

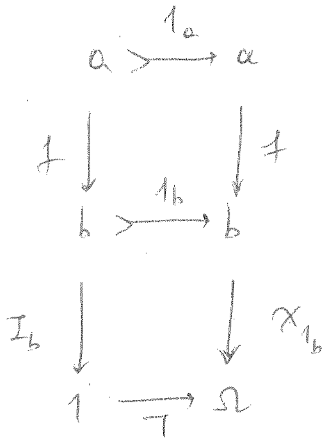
$\chi_{1_b} = \chi_{1_b} \circ 1_b = T \circ I_b = T_b$, since the inner square is pullback ($\Rightarrow$ comm)

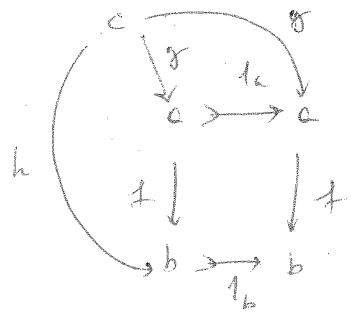
but we need to know what we want to show: $T_b \circ f = \chi_{1_b} \circ f = T \circ I_a = T_a$

So, it cannot be shown this way. We need to work as follows:

Consider the diagram: The bottom square is by Ω-axiom a pullback.

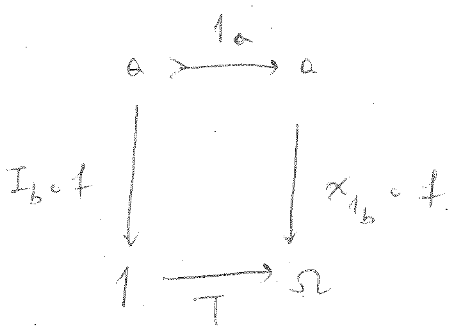
The upper square is also pullback.

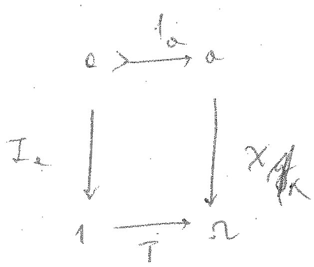




$f \circ 1_a = 1_b \circ f = f$. If $g: c \to a$, $h: c \to b$ st $f \circ g = 1_b \circ h = h$ then g is the unique arrow $c \to a$ st $1_a \circ g = g$ and $f \circ g = h$.

By PBL(i)





is pullback. Since $I_b \circ f = I_a$ (by uniqueness of $\to 1$ arrow), and since it is pullback by Ω-axiom we get $\chi_g = \chi_{1_b} \circ f$. Hence

$$
\begin{aligned}
(T \circ I_a =)\quad T_a &= \chi_{1_a} \circ 1_a \\
&= \chi_{1_a} \\
&= \chi_{1_b} \circ f \\
&= (\chi_{1_b} \circ 1_b) \circ f \\
&= (T \circ I_b) \circ f \\
&= T_b \circ f
\end{aligned}
$$

□

Section 2.3 Ω-axiom as a form of the Comprehension Axiom

This is a first step in the full answer to the MAIN question, that Topos sa- tisfies the axioms of ZF, appropriately interpreted. Of course, if it satisfies ZFC then it's going to be classical)

!!!

Motivation

If $\varphi : B \to 2$, then by Compr (Sep) there exists $A_\varphi = \{x \in B \mid \varphi(x) = 1\} = \varphi^{-1}(\{1\})$. Here we have the pullback

$$\begin{array}{ccc} A_\varphi & \hookrightarrow & B \\ {\scriptstyle !_{A_\varphi}} \downarrow & & \downarrow {\scriptstyle \varphi = \chi_{A_\varphi}} \\ 1 & \xrightarrow[\top]{} & 2 \end{array}$$

Definition 2.3.1 If $\mathcal{C}$ is a topos, and $1 \xrightarrow[\top]{} \Omega$ with $b \xrightarrow{\varphi} \Omega$, the pullback of this corner

$$\begin{array}{ccc} a & \xrightarrow{\{x:\varphi\}} & b \\ {\scriptstyle I_a} \downarrow & & \downarrow {\scriptstyle \varphi} \\ 1 & \xrightarrow[\top]{} & \Omega \end{array}$$

determines the (pseudo) subobject $\{x : \varphi\} : a \rightarrowtail b$ of b (by Prop. 1.?.?. $\{x:\varphi\}$ is monic, since $\top$ is always monic).

If $x : 1 \to b \in El(b)$ and $f : a \rightarrowtail b$ pseudo-subobject of b we define

$$\boxed{x \in f :\Leftrightarrow \exists_{\kappa : 1 \to a} \, (x = f \circ \kappa)}$$

$$\begin{array}{ccc} & 1 & \\ {\scriptstyle \kappa} \swarrow & & \searrow {\scriptstyle x} \\ a & \xrightarrow[f]{} & b \end{array}$$

here all are monics, i.e., κ is an element of a which f sends elements of a to elements of b.

Theorem ~~Proposition~~ 2.3.2 (Lawvere) Let $\mathcal{C}$ be a topos. Let $y : 1 \to b \in El(b)$. Then

$$\boxed{y \in \{x : \varphi\} \Leftrightarrow \varphi \circ y = \top} \qquad (\varphi \circ y = \text{true})$$

Proof

$$\begin{array}{ccc} 1 & \xrightarrow{y} & b \\ {\scriptstyle \kappa} \searrow & & \\ & a \xrightarrow{\{x:\varphi\}} b & \\ {\scriptstyle 1_1} \downarrow & {\scriptstyle I_a} \downarrow \quad \downarrow {\scriptstyle \varphi} & \\ & 1 \xrightarrow[\top]{} \Omega & \end{array}$$

($\Leftarrow$) Let $\varphi \circ y = \top$; then by the univ. property of the initial pullback we get

$$y = \{x:\varphi\} \circ \kappa \overset{\text{def}}{\Rightarrow} y \in \{x : \varphi\}$$

($\Rightarrow$) Let $y \in \{x:\varphi\} \overset{\text{def}}{\Rightarrow} \exists_{\kappa : 1 \to a} \left(y = \{x:\varphi\} \circ \kappa \right)$

But then
$$\begin{aligned} \varphi \circ y &= \varphi \circ (\{x:\varphi\} \circ \kappa) \\ &= (\varphi \circ \{x:\varphi\}) \circ \kappa \\ &= (\top \circ I_a) \circ \kappa = \top \circ (I_a \circ \kappa) = \top. \end{aligned}$$

Section 2.4 Examples of Topoi

Definition 2.4.1

- A **topos** is a category with 1, pullbacks, exponentials, and subobject classifier

(we use the equivalence: finitely complete ⟺ it has 1 and pullbacks.)
Th 1.3.2

Example 2.4.2 Set is a topos

Finset is a topos with limits, exp and $T: 1 \to \Omega$ as in Set

Example 2.4.3 $Set^2 = Set \times Set$ is a topos

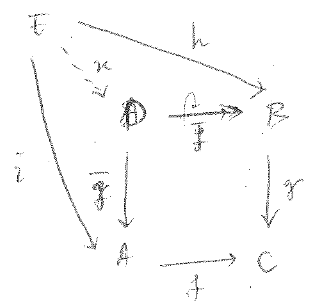
$(A,B) : Ob_{Set^2}$ $\quad (f,g) : (A,B) \to (C,D) \Leftrightarrow f : A \to C$ and $g : B \to D$

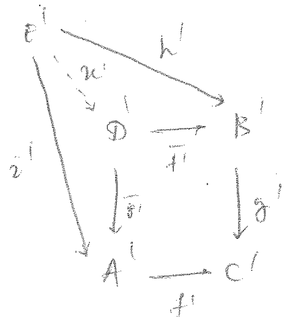
$1_{(A,B)} = (1_A, 1_B)$, $\quad (f,g) \circ (f', g') = (f \circ f', g \circ g')$

- $\underline{1_{Set^2}} = (1,1)$ $\quad (f,g) : (A,B) \to (1,1) \Leftrightarrow f : A \to 1$ and $g : B \to 1 \Leftrightarrow$
  $\Leftrightarrow f = I_A$, and $g = I_B$

- pullbacks in Set^2: Let a corner $(A,A') \xrightarrow[f']{f} (C,C')$ in Set^2, with $(B,B') \xrightarrow{g,\ g'} (C,C')$

Let the pullbacks

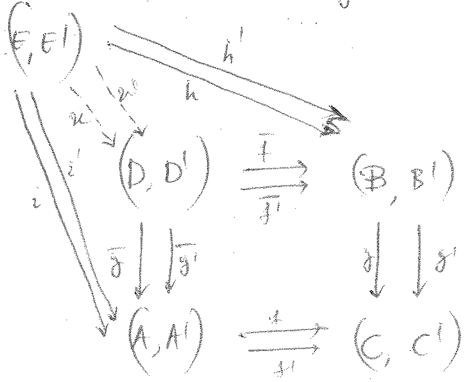




$g \circ \bar{f} = f \circ \bar{g}$ $\qquad g' \circ \bar{f}' = f' \circ \bar{g}'$

$g \circ h = f \circ i \Rightarrow \exists! u : E \to A$ $\qquad g' \circ h' = f' \circ i' \Rightarrow \exists! u' : E' \to D$ st.

$\bar{f} \circ u = h$, $\bar{g} \circ u = i$ $\qquad \bar{f}' \circ u' = h'$, $\bar{g}' \circ u' = i'$



$(g, g') \circ (\bar{f}', \bar{f}) = (g \circ \bar{f}', g' \circ \bar{f}) =$
$= (f \circ \bar{g}, f' \circ \bar{g}')$
$= (f, f') \circ (\bar{g}, \bar{g}')$

If $(h,h') : (E,E') \to (B,B')$ and $(i,i') : (E,E') \to (A,A')$ s.t.

$$(g,g') \circ (h,h') = (f,f') \circ (i,i') \Leftrightarrow ((g \circ h), g' \circ h') = (f \circ i, f' \circ i')$$

$$\Leftrightarrow g \circ h = f \circ i \text{ and } g' \circ h' = f' \circ i'$$

Then $(u,u') : (E,E') \to (D,D')$ s.t.

$$(\bar{f}, \bar{f}') \circ (u,u') = (\bar{f} \circ u, \bar{f}' \circ u') = (h,h') \text{ and}$$

$$(\bar{g}, \bar{g}') \circ (u,u') = (\bar{g} \circ u, \bar{g}' \circ u') = (i,i').$$

Uniqueness of (u,u') follows immediately by the uniqueness of u, u'.

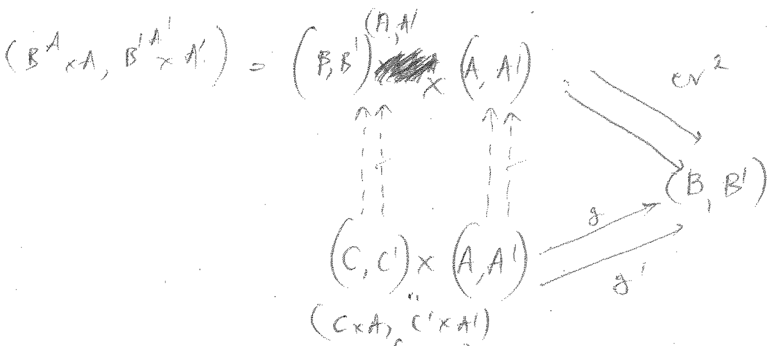
- Exponentials: $(A,A'), (B,B') \in Set^2$

We define $(B,B')^{(A,A')}$ and

$$ev^2 : (B,B')^{(A,A')} \times (A,A') \rightrightarrows (B,B')$$

such that for every $(C,C') \times (A,A') \overset{g}{\underset{g'}{\rightrightarrows}} (B,B')$,

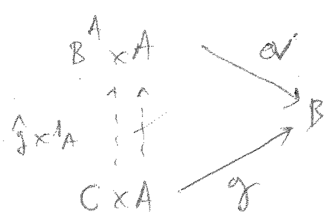
there is unique $\widehat{(g,g')} : (C,C') \to (B,B')^{(A,A')}$ s.t. the

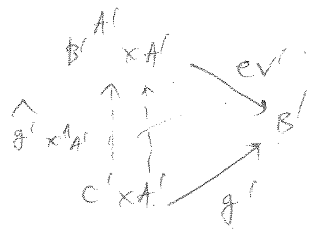


Define $(B,B')^{(A,A')} := (B^A, B'^{A'})$ and $(C,C') \times (A,A') = (C \times A, C' \times A')$

Hence we have $g : C \times A \to B$ and $g' : C' \times A' \to B'$

and unique $\hat{g} : C \to B^A$, $\hat{g}' : C' \to B'^{A'}$ s.t.





Hence we define $\widehat{(g,g')} \equiv (\hat{g}, \hat{g}')$ and it follows by the commutativity of the small diagrams that $ev^2 \circ \left((\hat{g}, \hat{g}') \times 1_{(A,A')} \right) = (g, g')$

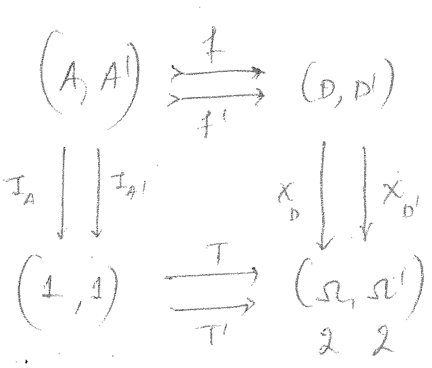
$$1_{(A,A')} = (1_A, 1_{A'})$$

$$= (\hat{g} \times 1_A, \hat{g}' \times 1_{A'})$$

$$\cdot \left(ev \circ (\hat{g} \times 1_A),\ ev' \circ (\hat{g}' \times 1_{A'}) \right).$$

Subobject classifier

- $T^2 : (1,1) \underset{T}{\overset{T}{\rightrightarrows}} (\Omega_2, \Omega_2)$



Corollary 2.4.4 If $\mathcal{C}_1$, $\mathcal{C}_2$ are topoi, then $\mathcal{C}_1 \times \mathcal{C}_2$ is a topos.

Example 2.4.5: $Set^{\rightarrow}$ is a topos. (the cat of functions)

Proof

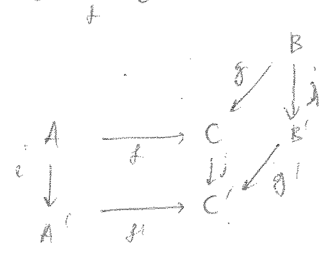
$$A \xrightarrow{f} B \qquad C \xrightarrow{g} D \qquad C' \xrightarrow{h} D' : Set^{\rightarrow} \qquad 1_f = (1_A, 1_B)$$

$$g \circ h = i \circ f \qquad \begin{array}{ccccc} A & \xrightarrow{h} & C & \xrightarrow{u} & \\ B & \xrightarrow{i} & D & \xrightarrow{e} & \end{array} \qquad Arr_{Set^{\rightarrow}}(f, g) \qquad \text{composition } (u,e)\circ(h,i) = (u\circ h,\ e\circ i)$$

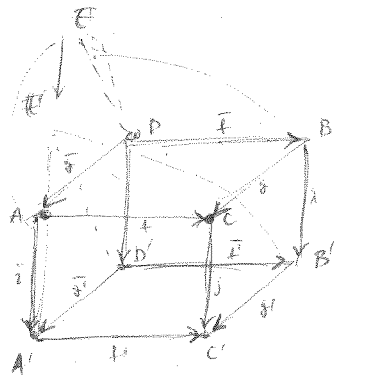
- $1_{Set^{\rightarrow}} \equiv \quad 1 \xrightarrow{1_1 = I_1} 1$

$$\begin{array}{c} A \\ f\downarrow \\ B \end{array} \longrightarrow \begin{array}{c} 1 \\ \downarrow 1 \\ 1 \end{array} \iff \begin{array}{ccc} A & \xrightarrow{I_A} & 1 \\ f\downarrow & & \downarrow 1_1 \\ B & \xrightarrow{I_B} & 1 \end{array} \qquad I_A = I_B \circ f \text{ necessarily by the uniqueness of } I_A.$$

- **Pullback** A corner $a \xrightarrow{f} c \xleftarrow{g} b$ in $Set^{\rightarrow}$ is a commut. diagrams



We want to define $D \xrightarrow{F} D'$ s.t.

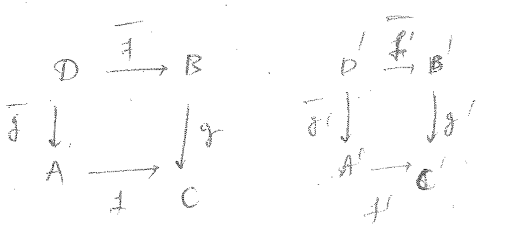


$(g,g')\circ(\bar f,\bar f') = (f,f')\circ(\bar g,\bar g') \Leftrightarrow$

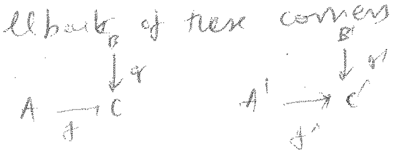
$(g\circ\bar f, g'\circ\bar f') = (f\circ\bar g, f'\circ\bar g') \Leftrightarrow$

$g\circ\bar f = f\circ\bar g$ and $g'\circ\bar f' = f'\circ\bar g'$

which suggests that the roof and the floor



need to be pullbacks. So we take D, D' to be the pullbacks of these corners

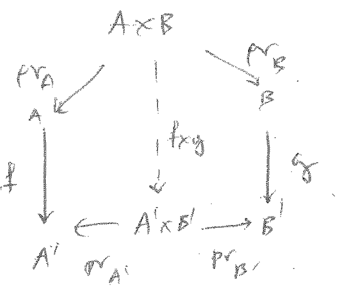


and the univ. property applied to $E \to E'$ follows from the univ property of E, and E'.

• <u>Exponentials</u> First we determine the product $\begin{smallmatrix}A\\ f\downarrow\\ A'\end{smallmatrix} \times \begin{smallmatrix}B\\ \downarrow g\\ B'\end{smallmatrix}$ in $Set^{\to}$

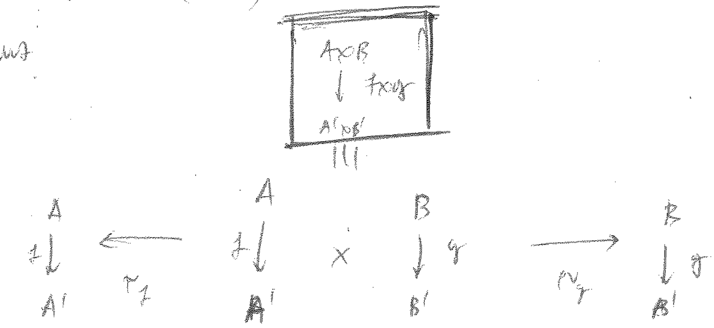
Recall that if $f: A\to A'$, $g: B\to B'$, then

$f\times g : A\times B \to A'\times B'$ is defined

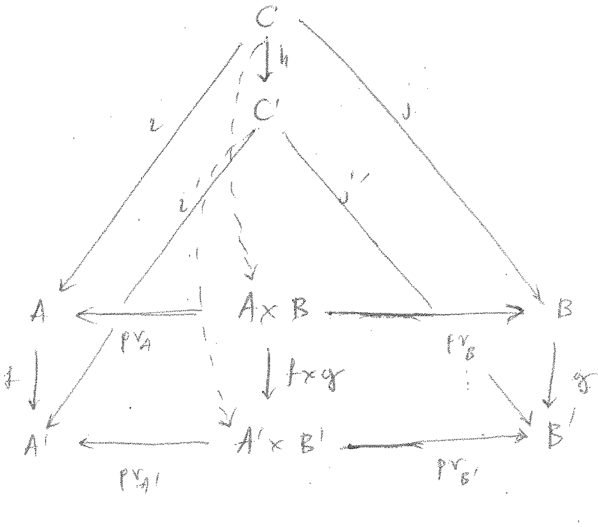


Hence $pr_{A'}\circ(f\times g) = f\circ pr_A$ and $pr_{B'}\circ(f\times g) = g\circ pr_B$

We want



So, we chose



the commutativity of this subdiagram follows from the aforementioned property of the f×g

cat the univ property of $A \times B \xrightarrow{f\times g} A' \times B'$

the transformations $C \to A \times B$ and $x' : C' \to A' \times B'$ follow from the univ. property of $A \times B$ and $A' \times B'$, i.e., $x = \langle i, j \rangle$ and $x' = \langle i', j' \rangle$.

The definition of $\begin{pmatrix} B \\ \downarrow g \\ B' \end{pmatrix}^{A \overset{f}{\downarrow} A'}$ is non-trivial (the obvious candidates do not work, special case of functor category theorem

"for a small cat $\mathcal{C}$ the functor cat $Set^{\mathcal{C}}$ all functors is a topos"

i.e. its class of objects, automorphisms is a set, not proper class.

It is the object

$$\begin{array}{c} E \\ \big\downarrow g^f \\ B'^{A'} \end{array}$$

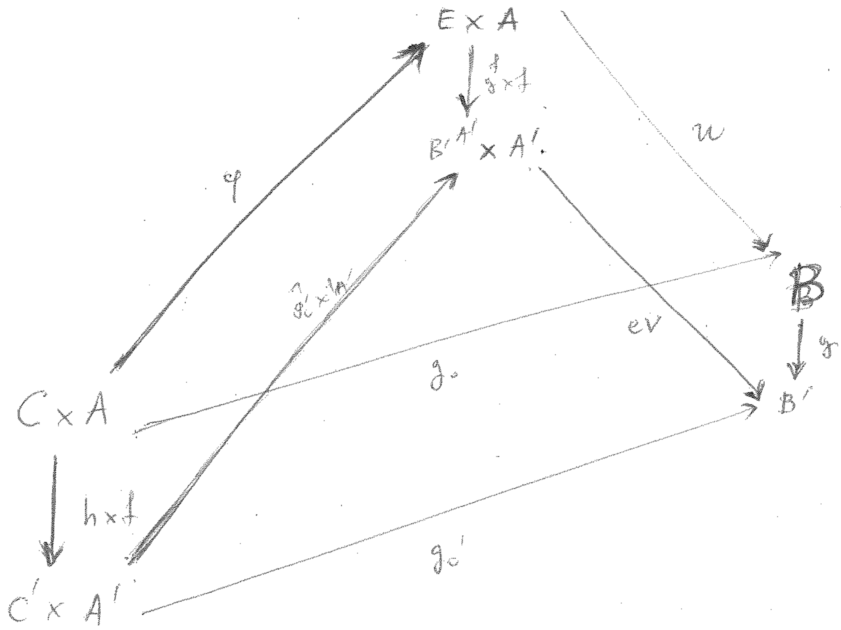
in $Set^{\to}$

where $E = \left[(x, x') \;\middle|\; \begin{array}{ccc} A & \xrightarrow{x} & B \\ f\big\downarrow & & \big\downarrow g \\ A' & \xrightarrow[x']{} & B' \end{array} \text{ commutes} \right]$

$$g^f\big((x, x')\big) = x'$$

Let $\begin{array}{c} C \\ \big\downarrow h \\ C' \end{array}$ and $\begin{array}{ccc} C \times A & \xrightarrow{g_0} & B \\ \big\downarrow h\times f & & \big\downarrow g \\ C' \times A' & \xrightarrow[g_0']{} & B' \end{array}$ st. $g \circ g_0 = g_0' \circ (h \times f)$.

We need to determine the evaluation map:



ev is the evaluation map corresponding to $B'^{A'}$, and

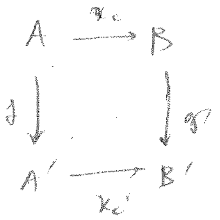
$u : E \times A \to B$ is defined by $u((u, u'), x) = u(x)$

We define $\varphi : C \times A \to E \times A$

$$\varphi(c, x) = ((u_c, u_c'), x)$$

where $u_c : A \to B$ is defined by $u_c(x) = g_0(c, x)$, $\forall x \in A$.

and



we define u_c' by the condition

$$u_c'(f(x)) = g'(u_c(x))$$

, (and arbitrarily if $y \in A' \setminus \mathrm{Im} f$. (It matters $B' \neq \emptyset$. If $B' = \emptyset$, there is no function $B \to B'$!!))

Hence, we get $u(\varphi(c, x)) = u((u_c, u_c'), x)$

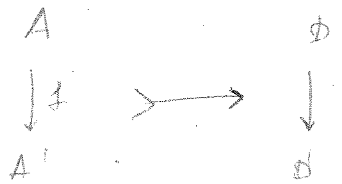
$= u_c(x)$

$= g_0(c, x)$

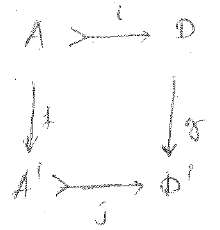
and we get the required commutativity of the (upper) inner triangle.

We also need to check that $(g^f \times f) \circ \varphi = (\hat{g}_0' \times 1_{A'}) \circ (h \times f)$ . . . .

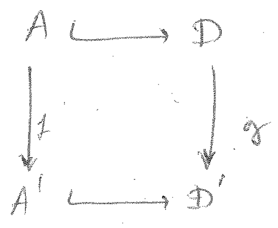
• Subobject classifier: Let



a monic arrow, i.e., it is immediate to see that this amounts to the existence of a pair of monics i, j with that [illegible]



Identifying A with $i(A)$ and A' with $j(A')$ we can w.l.g. take i, j to be inclusions



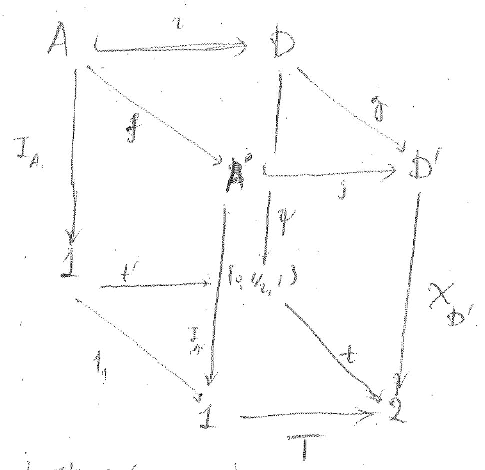
Hence if $x \in A$: $g(x) = f(x)$

- Let $x \in D$. Then
  - $x \in A$ $\vee$
  - $x \notin A$ and $g(x) \in A'$ $\vee$
  - $x \notin A$ and $g(x) \notin A'$.

Let $\psi : D \longrightarrow \{0, \frac{1}{2}, 1\}$

$$\psi(x) = \begin{cases} 1, & x \in A \\ 1/2, & x \notin A \text{ and } g(x) \in A' \\ 0, & x \notin A \text{ and } g(x) \notin A' \end{cases}$$

We form the cube



where $T(0) = 1$, $t'(0) = 1$ ⇒ the $A \to D$ … is commuting

$t : \{0, 1/2, 1\} \longrightarrow 2$

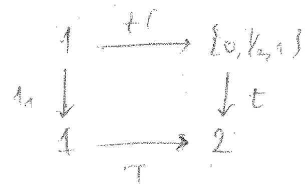
$t(0) = 0$, $t(\frac{1}{2}) = 1 = t(1)$

- The front and back (back is (composite)) faces are pullbacks in Set
- $(\psi, \chi_{D'}) \equiv \chi_{(i,j)}$

- The subobject classifier is the object

$$\Omega_{Set^{\to}} \equiv \{0, 1/2, 1\} \xrightarrow{\ t\ } 2$$

- and the true of $Set^{\to}$ is the arrow (t', T) $\top_{Set^{\to}}$



Definition 2.9.6

An element $x : 1 \to \Omega$ is called a *truth value* in the topos $\mathcal{C}$ (Ex?)

- In Set $x : 1 \to 2$ $\nearrow \bar{0}$, $\searrow \bar{1}$ two truth-values — Boolean algebra

- In Set^2 $x : (1,1) \underset{x_2}{\overset{x_1}{\rightrightarrows}} (2,2) \Leftrightarrow x_1 : 1 \to 2$ and $x_2 : 1 \to 2$

  Hence we have 4 truth values $(\bar{0}, \bar{0})$, $(\bar{0}, \bar{1})$, $(\bar{1}, \bar{0})$, $(\bar{1}, \bar{1})$ — Boolean algebra

- In $Set^{\to}$ the truth values are arrows between $(x, 1)$ Ω. t fcts

  $t(0) = 0$ $t(1/2) = t(1) = 1$

$$\begin{array}{ccc} 1 & \xrightarrow{x} & \{0, 1/2, 1\} \\ {\scriptstyle 1_1}\downarrow & & \downarrow t \\ 1 & \xrightarrow[e]{} & \{0,1\} \end{array}$$

$1 = \{0\}$

- $x(0) = 0$ $t(0) = 0$, take $e(0) = 0$ — $(\bar{0}, \bar{0})$
- $x(0) = 1/2$ $t(1/2) = 1$, take $e(0) = 1$ — $(\bar{1/2}, \bar{1})$
- $x(0) = 1$ — // — $(\bar{1}, \bar{1})$

So we have 3 truth values!

not a Boolean algebra, since it has 3 elements

(Heyting)

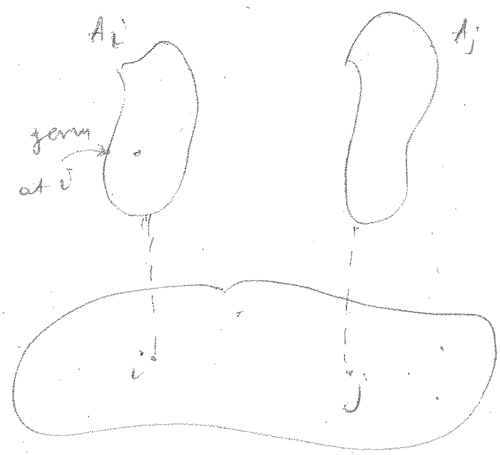
The category of bundles

- A **bundle** is the underlying set-theoretic structure of the **sheaf** concept. The primary source of TT was the study of sheaves in alg. geometry (Grothendieck).

Definition.

$$\mathcal{A} = \{A_i \mid i \in I\}$$
$$A_i \cap A_j = \emptyset, \quad i \neq j$$

is a bundle of sets over I



stalk space of the bundle — $A = \bigcup_{i \in I} A_i = \{x \mid \exists_{i \in I} (x \in A_i)\}$

$p : A \to I$

$p(x) = i$ s.t. $x \in A_i$ (i is unique by the disjointness of A_i's)

$A_i = p^{-1}(\{i\})$: the fiber over i (or the stalk)

A_i can be $\emptyset$. (If not, then p is onto I)

- Conversely, if $p : A \to I$ then (fiber structure is not necessarily surjection)

$A_i = p^{-1}(\{i\})$

$\mathcal{A} = \{A_i \mid i \in I\}$ is a bundle of sets over I.

- So a bundle of sets over I is just a function $p : A \to I$ "essentially"

- We define $\mathbf{Bn}(I) = \mathbf{Set}\downarrow I$ with objects $f : A \to I$

To prove that this is a topos is a special case of the fund. theorem for topoi.

A sheaf is a bundle with additional topological structure.

Definition A <u>sheaf over I</u>, where (I, θ) is a top. space, is a pair

$$(A, p) \quad \text{where}$$

(i) (A, t) is a top. space

(ii) $p : A \to I$ is a continuous function which is a <u>local homeomorphism</u>

i.e., for every $x \in A$, there is open $U \in t$ s.t.

(a) $x \in U$

(b) $p|_U$ is a homeomorphism $\left(p|_U : U \to p(U)\right)$

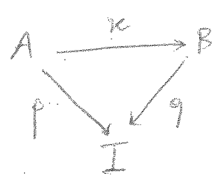
(c) $p(U)$ is open I

Definition

- Category $Top(I)$ of sheaves over I has

<u>objects</u>: sheaves (A, p)

<u>arrows</u>: $k : (A, p) \to (B, q)$, where $k : A \xrightarrow{\text{cont}} B$ s.t. $+$ $f.d.c$



<u>Remark</u>
- k turns out to be local homeomorphism
- k turns out to be open function (maps open in A to open in B)
- $k(A)$ open in B.

<u>$Top(I)$ is a topos</u> (a spatial topos)

Main Observation

All our defn – constructions – proofs (at the categorical level)

so far are constructive.

To do some proofs in Set we need cl logic

$x \in A \vee x \notin A$ eg

but the abstraction in cat Th. involved only constr. arguments.

So, it is not a ~~coincidence~~ whim that the logic of a topos is in general intuitionistic. !!!

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topos model of HoTT

Cats + Types : some comments for the ~~S~~ students