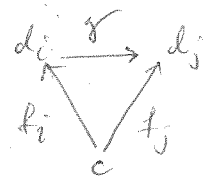
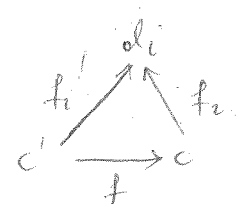


Section 1.7 Limits and co-limits

- Definition 1.7.1 • A diagram \mathcal{D} in \mathcal{C} is a collection of \mathcal{C} -objects $(d_i)_{i \in I}$ with some arrows $(f_{ij}) : d_i \rightarrow d_j$. (possibly more than one arrow $d_i \rightarrow d_j$, possibly none)
- A cone for diagram \mathcal{D} is $(c, (f_i)_{i \in I})$, where $c \in \text{Ob } \mathcal{C}$ and $f_i : c \rightarrow d_i$ $\forall i \in I$ s.t. $f_i = g \circ f_j$ for $d_i \xrightarrow{g} d_j$ in \mathcal{D} .



- A limit for a diagram \mathcal{D} is a \mathcal{D} -cone $(f_i : c \rightarrow d_i)_{i \in I}$ s.t. if $(f'_i : c' \rightarrow d_i)_{i \in I}$ is any other \mathcal{D} -cone, there is unique $f : c' \rightarrow c$ s.t. $f_i = f \circ f'_i$.

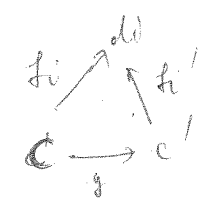


for every d_i in \mathcal{D} . ("any other cone factors through the limit")

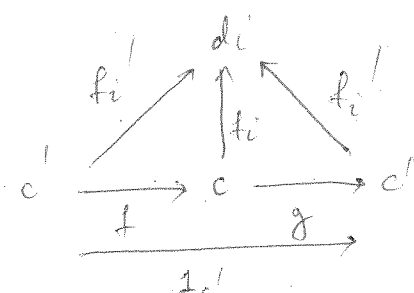
Remark 1.7.2 A limit for \mathcal{D} is unique up to isomorphism.

Proof: (this proof is a generalization of the uniqueness of the product up to iso)

Let c' with the same universal property.



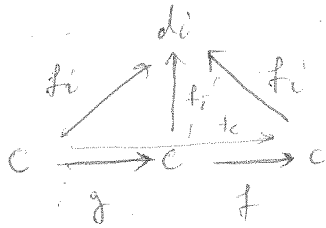
We have that



$$f'_i \circ (g \circ f) = (f'_i \circ g) \circ f = f_i \circ f = f'_i$$

By uniqueness of the arrow between c and c' , and since 1_c has this property too, we get $g \circ f = 1_c$.

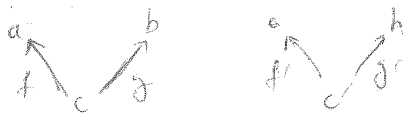
Similarly for the diagram



we get $f \circ g = tc$. Hence $c \cong c'$. \square

Example 1.7.3 (i) Let $D = [a, b]$ the diagram with no arrows. $a \quad b$

A D -cone is $(c, f: c \rightarrow a, g: c \rightarrow b)$



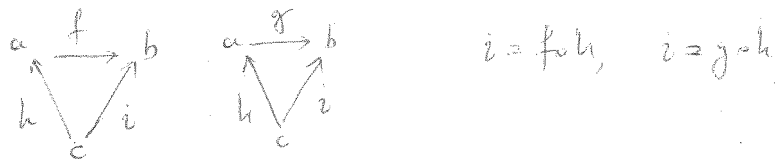
A limit for D is a D -cone that for every other D -cone $(c', f': c' \rightarrow a, g': c' \rightarrow b)$ there is a unique e s.t.



i.e. c is a product of a, b in C .

(ii) $D = [a, b]$ and $f, g: a \rightarrow b$ i.e. $a \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} b$ is our diagram

A D -cone is $(c, h: c \rightarrow a, i: c \rightarrow b)$ \square



i.e. $c \begin{smallmatrix} \xrightarrow{h} \\ \xrightarrow{i} \end{smallmatrix} a \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} b$ $f \circ h = g \circ h$

A D -limit is an equalizer.

(iii) $D = [\]$ the empty diagram

A D -cone is a C -object c .

A limit for D is c i.e. for every other $c': C$ there is $f: c' \rightarrow c$, i.e. c is terminal.

Ex: Define co-cone for D

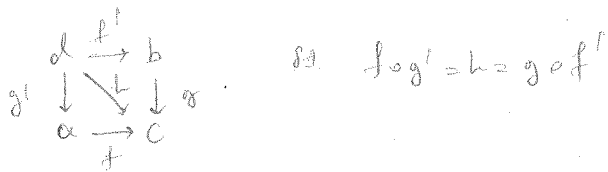
co-limit for D .

a, b, c, d co-objects, co-egs, initial objects

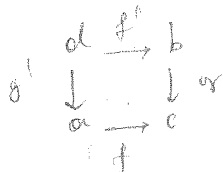
Section 1.8 Pullbacks and pushouts

Definition 1.8.1 Let the diagram $D: a \xrightarrow{f} c \begin{matrix} \downarrow \gamma \\ b \end{matrix}$ in \mathcal{C} .

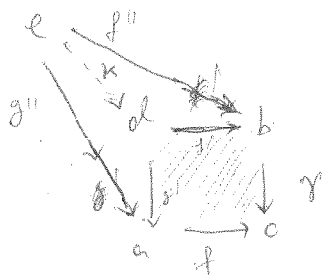
A cone for D is $(d, g: d \rightarrow a, h: d \rightarrow c, f' = d \rightarrow b)$



Hence a cone is a pair $(d, g': d \rightarrow a, f' = d \rightarrow b)$ s.t. the square commutes



A pullback of $a \xrightarrow{f} c \begin{matrix} \downarrow \gamma \\ b \end{matrix}$ is a limit for D (i.e., for every cone $(e, g'': e \rightarrow a, f'' = e \rightarrow b)$ there is unique $\kappa: e \rightarrow d$ s.t. $f' \circ \kappa = f''$ and $g' \circ \kappa = g''$)



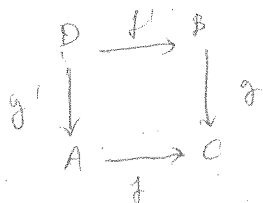
Let inner square (f', g', f'', g') is called pullback square (or Cartesian square)

We say that " f' arises by pulling back f along g' "

" g' arises by pulling back g along f "

It is the most important limit in TT.

Examples 1.8.2 (2) In Set the pullback of $A \xrightarrow{f} C \begin{matrix} \downarrow \gamma \\ B \end{matrix}$

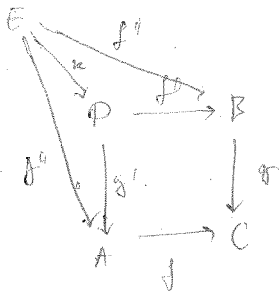


is defined by $D = \{ (x, y) \in A \times B \mid f(x) = g(y) \} \cong A \times_C B$ (the fiber product)

$$f': D \rightarrow B \quad f'(x, y) = y$$

$$g': D \rightarrow A \quad g'(x, y) = x$$

Prove the universal property of D .



$$\kappa: E \rightarrow D \subseteq A \times B$$

$\kappa(e) = (g''(e), f''(e))$, the only possibility from the data

κ is well-defined, since

$$f(f''(e)) = g(g''(e))$$

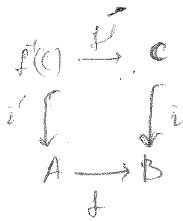
and $f \circ g'' = g \circ f''$ is part of our hypothesis for κ

$$\begin{aligned} \text{Clearly } f'(\kappa(e)) &= f'(g''(e), f''(e)) \\ &= f''(e) \end{aligned}$$

$$\begin{aligned} g'(\kappa(e)) &= g'(g''(e), f''(e)) \\ &= g''(e) \end{aligned}$$

By the same two conditions follow (copart from the def of κ) the uniqueness. \square

(ii) In Set, if $f: A \rightarrow B$ and $C \subseteq B$, then $f^{-1}(C) = \{x \in A \mid f(x) \in C\}$



$$A \times_C C = \{(x, y) \in A \times C \mid f(x) = i(y) = y\}$$

$$f'(x, y) = y = f(x)$$

$$i'(x, y) = x$$

Clearly: $A \times_B C \cong f^{-1}(C) \quad (x, y) \in A \times_B C \mapsto x$

$x \in f^{-1}(C) \mapsto (x, y)$, where y is the unique element of C s.t. $f(x) = y$.

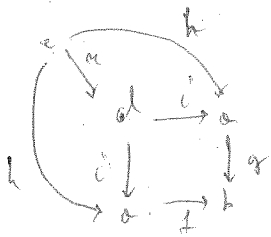
Hence $f^{-1}(C)$ arises by pulling C back along f .

(Categorical character of the concept of function, ^{needs} not deal with sets of ordered pairs)

Proposition 18.3

If $\begin{array}{ccc} d & \xrightarrow{i} & a \\ i' \downarrow & & \downarrow g \\ a & \xrightarrow{f} & b \end{array}$ is a pullback, then i is an equalizer of f, g

Proof:

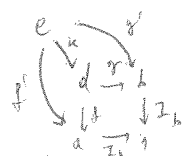


can be written $\begin{array}{ccc} d & \xrightarrow{i} & a \\ \uparrow i' & & \uparrow h \\ c & & c \end{array} \xrightarrow{f} b \quad \text{is } \text{is equal}(h, g)$
 $\therefore \kappa = h$

Proposition 18.4

If e has terminal 1 and $\begin{array}{ccc} d & \xrightarrow{g} & b \\ f \downarrow & & \downarrow I_b \\ a & \xrightarrow{I_a} & 1 \end{array}$ is a pullback, then d is a product of a, b

Proof:



$I_b \circ g = I_b \circ f'$, since $d \rightarrow 1$ is unique. Similarly $I_a \circ f' = I_a \circ g'$. Hence the conditions are satisfied that d is a product.

Proposition 1.8.5 (The pullback lemma \rightarrow PBL). Let the following diagram commute

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ h \downarrow & & \downarrow g \\ d & \xrightarrow{e} & c \\ j \downarrow & & \downarrow i \\ a' & \xrightarrow{k} & b' \end{array}$$

(i) If the two small squares are pullbacks, then the outer rectangle $\begin{array}{ccc} a & \xrightarrow{f} & b \\ j \downarrow & & \downarrow i \\ a' & \xrightarrow{k} & b' \end{array}$ is a pullback.

(ii) If the outer rectangle and the bottom square are pullbacks, then the top square is a pullback.

(It is used in the proof of uniqueness of the sub-object classifier)

Proposition 1.8.6 $f: a \rightarrow b$ is monic $\Leftrightarrow \begin{array}{ccc} a & \xrightarrow{1_a} & a \\ 1_a \downarrow & & \downarrow f \\ a & \xrightarrow{f} & b \end{array}$ is a pullback square

Proof: let $c \begin{array}{c} \xrightarrow{r} \\ \xrightarrow{h} \end{array} a \xrightarrow{f} b$ s.t. $f \circ g = f \circ h \rightarrow g = h$

Clearly $f \circ 1_a = f \circ 1_a = f$. Let c, h, g s.t. $f \circ h = f \circ g$. By hypothesis $h = g$.

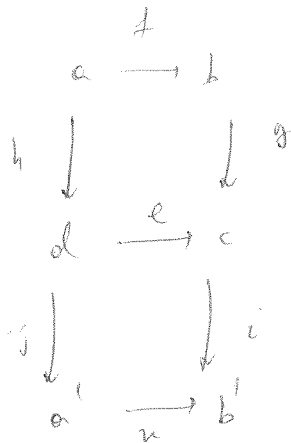
Let $\begin{array}{ccc} c & & h \\ \downarrow r & \searrow k & \downarrow 1_a \\ a & \xrightarrow{f} & a \\ \downarrow 1_a & & \downarrow f \\ a & \xrightarrow{f} & b \\ & & \downarrow f \end{array}$ Define $k: c \rightarrow a$ $u = h$. Then $1_a \circ u = h = g$.

If it is a pullback square then it's immediate that $h = 1_a \circ u = g$.

Proposition 1.8.7 If $\begin{array}{ccc} a & \xrightarrow{f} & b \\ i \downarrow & & \downarrow j \\ c & \xrightarrow{g} & d \end{array}$ is a pullback square and g is monic, then f is monic.

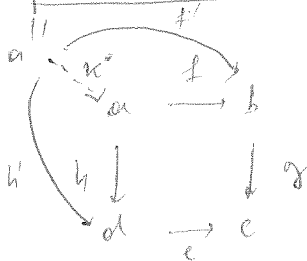
$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ i \downarrow & & \downarrow j \\ c & \xrightarrow{g} & d \end{array}$$

(9)



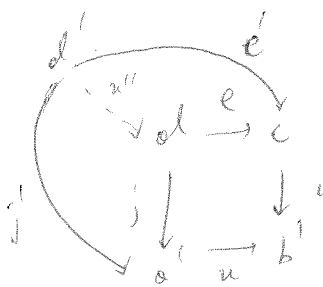
Hypothesis

$$\begin{array}{l}
 g \circ f = e \circ h \quad (1) \\
 i \circ e = u \circ j \quad (2)
 \end{array}$$



~~g \circ f' = e \circ h~~ (I)

$g \circ f' = e \circ h'$
 $\exists ! u' (f \circ u' = f' \text{ and } h \circ u' = h')$



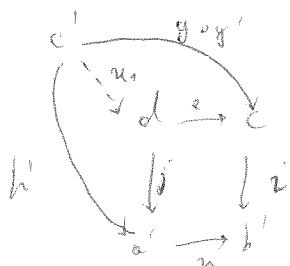
$i \circ e' = u \circ j'$
 $\exists ! u'' (e \circ u'' = e' \text{ and } j \circ u'' = j')$ (II)

First we show that $(i \circ j) \circ f = u \circ (j \circ h)$

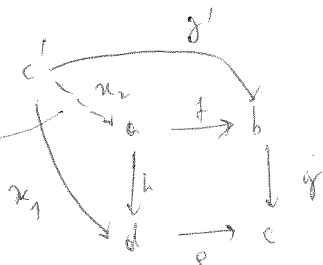
$$\begin{aligned}
 (i \circ j) \circ f &= i \circ (j \circ f) \\
 &\stackrel{(1)}{=} i \circ (e \circ h) \\
 &= (i \circ e) \circ h \\
 &\stackrel{(2)}{=} (u \circ j) \circ h \\
 &= u \circ (j \circ h)
 \end{aligned}$$

Let $d' \in (i \circ j) \circ y'$ and h', g'

$$(i \circ j) \circ y' = u \circ h'$$



By (II) there is $u_1: c' \rightarrow a'$ s.t. $e \circ u_1 = g \circ g'$ and $j \circ u_1 = h'$



Further show that $g \circ g' = e \circ u_1$, which is just

By (I) there is $u_2: c' \rightarrow a'$ s.t.

$$f \circ u_2 = g' \text{ and } h \circ u_2 = u_1$$

We need to show $f \circ u_2 = g'$, which is

$$\text{and } (j \circ h) \circ u_2 = h'$$

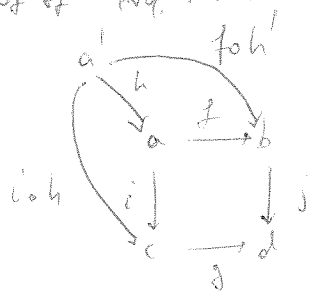
$$(j \circ h) \circ u_2 = j \circ (h \circ u_2)$$

$$= j \circ u_1$$

z_1 , by the def of u_1 .

(ii) Similarly.

Proof of Prop. 1.1.7.



$$f \circ h = f \circ h'$$

$$j \circ f = j \circ i$$

$$\begin{aligned} j \circ (f \circ h) &= j \circ (f \circ h') \\ &= (j \circ f) \circ h \\ &= (j \circ i) \circ h \\ &= j \circ (i \circ h) \end{aligned}$$

$$\exists! k: (f \circ u = f \circ h')$$

$$i \circ u = i \circ h$$

$$\text{Since } f \circ h' = f \circ h$$

$$f \circ h = f \circ h$$

$$i \circ h = i \circ h$$

$$i \circ h' = i \circ h ?$$

$$\begin{aligned} j \circ (i \circ h') &= (j \circ i) \circ h' = (j \circ f) \circ h' \\ &= j \circ (f \circ h') \\ &= j \circ (f \circ h) \\ &= (j \circ f) \circ h \\ &= (j \circ i) \circ h \\ &= j \circ (i \circ h) \end{aligned}$$

\Rightarrow unique

$$\Rightarrow i \circ h' = i \circ h$$

Hence by uniqueness of u , $h = h'$.

D

Section 1.9 Completeness and co-completeness

- Definition 1.1 \mathcal{C} is complete, if every diagram \mathcal{D} in \mathcal{C} has a limit in \mathcal{C}
- \mathcal{C} is co-complete, if " " " " " co-limit in \mathcal{C} .
- \mathcal{C} is bi-complete, if it is complete and co-complete.
- \mathcal{C} is finitely complete, if every finite diagram \mathcal{D} in \mathcal{C} (i.e., \mathcal{D} has finite number of objects and arrows between them) has a limit in \mathcal{C} .
- \mathcal{C} is finitely co-complete, if every finite diagram \mathcal{D} in \mathcal{C} has a co-limit in \mathcal{C} .
- \mathcal{C} is finitely bi-complete, if it is finitely complete and finitely co-complete.

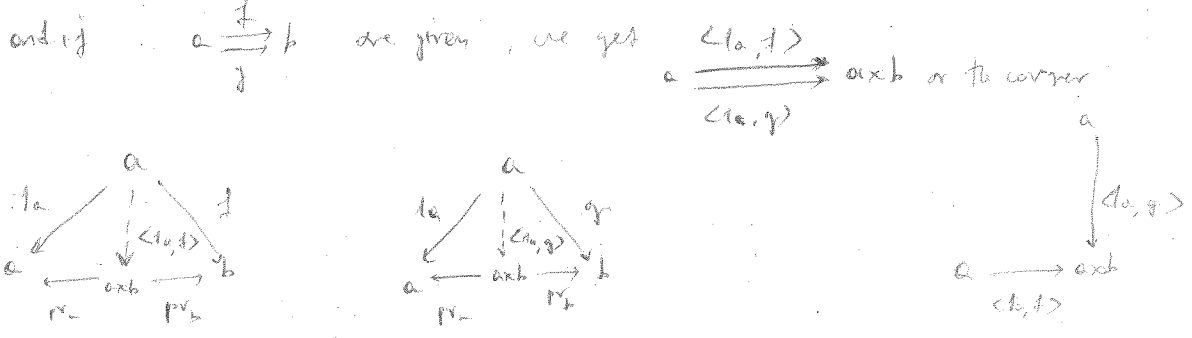
Theorem 1.2 If \mathcal{C} has a terminal object and a pullback for every pair of \mathcal{C} -arrows with common codomain $a \xrightarrow{f} c$, then \mathcal{C} is finitely complete.

(+ dual of Th. 1.2-2) (co =)
 i.e. finite completeness rests only on pullbacks (this is important), when a terminal object exists

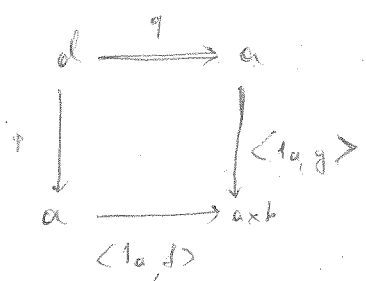
Proof: ~~not complete~~ not complete. 1 out pullback \Rightarrow a x b (Prop. 1.8.4) $\begin{matrix} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ c & \xrightarrow{g} & d \end{matrix}$
 pullback + product \Rightarrow equations (very Prop. 1.8.3)

Remark 1.2.3 The converse of Th. 1.2 holds trivially: the terminal object is the limit of the empty diagram and pullbacks were defined on limits.

⊙ Elaboration: Given pullbacks + products (which we get from the previous step)



Hence we get their pullback



Since the previous square commutes we have

$$\langle \iota_a, g \rangle \circ q = \langle \iota_a, f \rangle \circ p$$

but by the obvious properties (given exp in the section of products)

$$\begin{aligned} \langle \iota_a, g \rangle \circ q &= \langle \iota_a \circ q, g \circ q \rangle = \langle q, g \circ q \rangle \\ \langle \iota_a, f \rangle \circ p &= \langle \iota_a \circ p, f \circ p \rangle = \langle p, f \circ p \rangle \end{aligned} \quad \Bigg] \Rightarrow p = q \text{ and } f \circ p = g \circ p$$

Hence the pullback is written as

$$\begin{array}{ccc} d & \xrightarrow{p} & a \\ p \downarrow & & \downarrow \langle \iota_a, g \rangle \\ a & \xrightarrow{\langle \iota_a, f \rangle} & a \times b \end{array}$$

and by Prop 1.8.3, $d \xrightarrow{p} a$ is an equalizer of $\langle \iota_a, g \rangle$ and $\langle \iota_a, f \rangle$

(If $d \xrightarrow{c} a$
 $\downarrow \quad \downarrow$
 $a \quad b$
 is pullback,
 then $ac = bc$)

$$\begin{array}{ccc} d & \xrightarrow{p} & a \\ \uparrow x & \nearrow h & \\ c & & a \times b \end{array} \begin{array}{l} \langle \iota_a, f \rangle \\ \langle \iota_a, g \rangle \end{array}$$

since $\langle \iota_a, f \rangle \circ p = \langle \iota_a, g \rangle \circ p \Leftrightarrow f \circ p = g \circ p$

and similarly for h, x ,

we get that $d \xrightarrow{p} a$ is also an equalizer of f, g . \square

• Remark in Makkai 84, 37, If \mathcal{C} is finitely complete then \mathcal{C}^{fin} is finitely complete.

but if \mathcal{C} has exponentials, then \mathcal{C}^{fin} may not (that they do is an essential property of \mathcal{C}^{ccc} , it's not why we put this into the definition)

Section 1.10 Exponentials

Definition 1.10.1 In Set, if A, B : Set, then $B^A = \{f: A \rightarrow B\}$

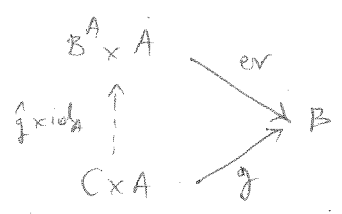
We define $ev: B^A \times A \rightarrow B$

$$ev(f, x) = f(x),$$

which is called the evaluation function on B^A (or $ev_{A,B}$)

Proposition 1.10.2: For every $g: C \times A \rightarrow B$

there is a unique $\hat{g}: C \rightarrow B^A$ s.t.



commutes, where $(\hat{g} \times id_A)(c, x) = (\hat{g}(c), id_A(x)) = (\hat{g}(c), x)$, $c \in C, x \in A$

Proof: Let C be set and $g: C \times A \rightarrow B$. From these data the only candidate $C \rightarrow B^A$ is

$$\hat{g}: C \rightarrow B^A$$

$$c \mapsto \hat{g}(c)$$

$$\hat{g}(c): A \rightarrow B$$

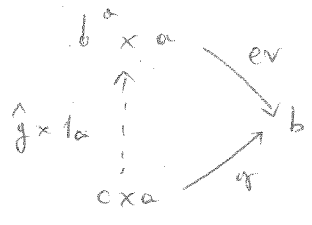
$$\hat{g}(c)(x) = g(c, x)$$

(F) Clearly, $ev(\hat{g} \times id_A(c, x)) = ev(\hat{g}(c), x) = \hat{g}(c)(x) = g(c, x)$

(I) If $h: C \rightarrow B^A$ is $ev(\hat{h} \times id_A(c, x)) = ev(\hat{h}(c), x) = \hat{h}(c)(x) = g(c, x)$

then $\hat{h}(c) = \hat{g}(c), \forall c \in C$.

Definition 1.10.3 A \mathcal{C} -mon has exponentials if for every $a, b \in \text{Ob } \mathcal{C}$ there is $b^a \in \text{Ob } \mathcal{C}$ and $ev: b^a \times a \rightarrow b$, an evaluation arrow (or ev_{ab}), s.t. for every $c \in \text{Ob } \mathcal{C}$ and $g: c \times a \rightarrow b$ there is unique $\hat{g}: c \rightarrow b^a$ s.t. $\hat{g} \cdot d = g$



$\hat{g} \cdot d = g$
 (read $\hat{g} \times g: c \times a \rightarrow b$
 $\hat{g} \times g = \langle \text{top } \hat{g}, g \circ \text{pr}_a \rangle$)

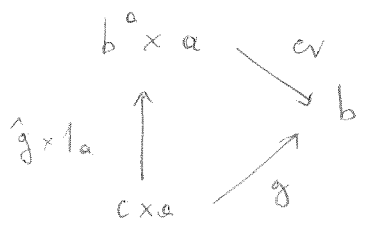
i.e., $ev \circ (\hat{g} \times id_a) = g$.

$$(b^a, ev: b^a \times a \rightarrow b)$$

(Ex) "Uniqueness of exponential up to isomorphism."

$$\forall c, ob_e \quad \forall g: c \times a \rightarrow b \quad \exists! \hat{g}: c \rightarrow b^a \quad \text{st.}$$

(I)

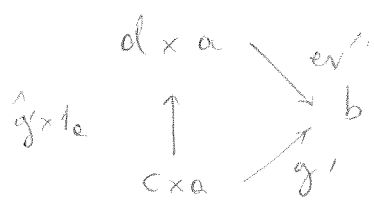


$$ev \circ (\hat{g} \times 1_a) = g$$

Let $(d, ev': d \times a \rightarrow b)$

$$\forall c, ob_e \quad \forall g': c \times a \rightarrow b \quad \exists! \hat{g}': c \rightarrow d$$

(II)

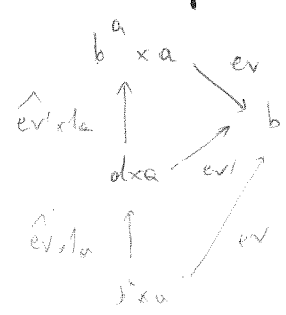


$$ev' \circ (\hat{g}' \times 1_c) = g'$$

d wants $f: d \xrightarrow{\leftarrow} b^a$ iso

From (I) take $c = d$

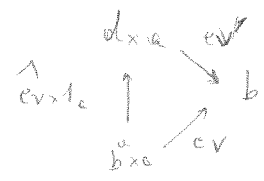
$$\begin{aligned}
 g &= ev': d \times a \rightarrow b \\
 \exists! \hat{ev}' &: d \rightarrow b^a
 \end{aligned}$$



outer diagram commutes
 $ev = (ev \circ \hat{ev}') \circ (ev' \times 1_d)$
 $= ev \circ (\hat{ev}' \times 1_d) \circ (ev' \times 1_d)$
 $= ev \circ [(\hat{ev}' \circ ev') \times 1_d]$
 Hence $\hat{ev}' \circ ev' = 1_d$
 Similarly $ev \circ \hat{ev}' = 1_{b^a}$

From (II) take $c = b^a$

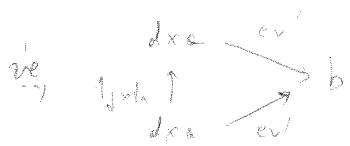
$$\begin{aligned}
 g' &= ev: b^a \times a \rightarrow b \\
 \exists! \hat{ev} &: b^a \rightarrow d
 \end{aligned}$$



$$d \xrightarrow{\hat{ev}'} b^a \xrightarrow{\hat{ev}} d$$

It suffices to show that $\hat{ev} \circ \hat{ev}'$ has the univ. prop. of ev for $c = d$ and $g' = ev'$, because then $\hat{ev} \circ \hat{ev}' = 1_d$, from!

$$\begin{aligned}
 & \cancel{(\hat{ev}' \circ ev')} \times 1_a \circ (\hat{ev}' \times 1_a) \circ (ev' \times 1_a) \stackrel{ex}{=} (\hat{ev}' \circ \hat{ev}') \times (1_d \times 1_a) \\
 & = (\hat{ev}' \circ \hat{ev}') \times 1_a
 \end{aligned}$$



Proposition 1.10.4 The function $\hat{A} = \text{Arr}_e(c \times a, b) \rightarrow \text{Arr}_e(c, b^a)$
 is a bijection $f \mapsto \hat{f}$

Proof: Let $g, h: c \times a \rightarrow b$ s.t. $\hat{g} = \hat{h}$. Then $\hat{g} \times 1_a = \hat{h} \times 1_a$, and $\text{ev} \circ (\hat{g} \times 1_a) = \text{ev} \circ (\hat{h} \times 1_a) \Leftrightarrow g = h$

Let $\varphi: c \rightarrow b^a$

Hence $\text{ev} \circ (\varphi \times 1_a): c \times a \rightarrow b$

We define $\hat{\varphi} = \text{ev} \circ (\varphi \times 1_a)$

Also, $\hat{\varphi} = \text{ev} \circ (\hat{\varphi} \times 1_a)$

By the uniqueness of \hat{g} we get $\hat{\varphi} \geq \varphi$, i.e. $\hat{\varphi}$ is a surjection.

(g, \hat{g}) are called exponential adjoints of each other.

Definition 1.10.5 A cartesian closed category is a finitely complete category with exponentiation ^{by}

Eg Set

Theorem 1.10.6 Let \mathcal{C} be a cartesian closed cat with an initial object 0

- (i) $0 \cong 0 \times a$, $a \cong 0^a$
- (ii) If $f: a \rightarrow 0$, then $a \cong 0$
- (iii) If $0 \cong 1$, then all \mathcal{C} -objects are isomorphic (\mathcal{C} is degenerate)
- (iv) If $f: 0 \rightarrow a$, then f is monic (The unique 0_a is monic)
- (v) $a^1 \cong a$, $a^0 \cong 1$, $1^a \cong 1$ (where 1 is the terminal object which exists by the hypothesis of finitely complete)

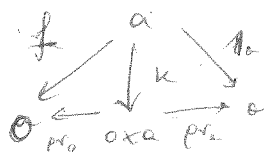
Proof (i) By Prop 1.10.4: $\text{Arr}_e(0 \times a, b) \cong \text{Arr}_e(0, b^a)$
 \parallel
 $[0_{ba}]$ by initiality of 0.

Let for every b , there is a unique arrow $0 \times a \rightarrow b$

I.e., $0 \times a$ is initial in \mathcal{C} , hence $0 \times a \cong 0$. (by Prop. 1.9.2)

(ii) We show that $a \cong 0 \times a$ hence by (i) $a \cong 0$.

By the universal property of the product there is unique x st.



$$\begin{aligned}
 \text{pr}_1 \circ x &= 1_a \\
 \text{pr}_0 \circ x &= f
 \end{aligned}$$

We need to show that $x \circ \text{pr}_a = 1_{0 \times a}$.

Since $x \circ \text{pr}_a : 0 \times a \rightarrow 0 \times a$, and by (i) $0 \times a$ is initial, there is a unique arrow $0 \times a \rightarrow 0 \times a$, hence $x \circ \text{pr}_a = 1_{0 \times a}$. \blacksquare

(iii) Let $\varphi : 1 \rightarrow 0$. Since $a \xrightarrow{1_a} 1 \xrightarrow{\varphi} 0$, $\varphi \circ 1_a = a \rightarrow 0$, we use (ii) to get $a \cong 0$, for every a . $a \cong 0 \cong b \Rightarrow a \cong b$.

(iv) Let $b \xrightleftharpoons[f]{f} 0 \xrightarrow{0_a} a$ and $0_a \circ f = 0_a \circ g$

Since there is an arrow $b \rightarrow 0$, by (ii) $b \cong 0$ and b is initial. Hence there is unique arrow $b \rightarrow a$. i.e., $f = g$.

(v) Exercise

Remark 1.10.7 In Set monic epis are iso (not true in Mon), but true in Grp, but Grp not cartesian closed, since $0 \cong 1$ but not degenerate

$$\begin{aligned}
 \varphi : \{0\} &\rightarrow G & \varphi(0) &= e \\
 \vartheta : G &\rightarrow \{0\} & \vartheta(g) &= 0, \forall g \in G
 \end{aligned}$$

$$\text{not } G_1, G_2 : \text{Grp} \quad G_1 \cong G_2$$

There are cart-closed sets with monic epis that are not iso (not).

If we need to find extra property (ie) that characterise Set. (subobject classifier).

Chapter II Banach Theory of Topol

Section 2.1 Subobjects

$$\mathcal{P}(B) = \{A \subseteq B\}$$

$(\mathcal{P}(B), \subseteq)$ p-set, hence \subseteq ext

chr $A_1 \xrightarrow{f} A_2 \Leftrightarrow A_1 \subseteq A_2$ and $f \text{ chr}$

$$\begin{array}{ccc} & A_2 & \\ & \uparrow & \searrow \\ & A_1 & \rightarrow B \end{array}$$

Motivation:

$A \subseteq B$ then $\text{id}_A: A \hookrightarrow B$ is injective (\Rightarrow monic).

If $f: A \rightarrow B$, then $\text{Im}(f) = \{f(x) \mid x \in A\} \subseteq B$. Actually $A \cong \text{Im}(f)$.

Definition 2.1.1 Let $b = \text{Ob } \mathcal{C}$. A pseudo-subobject of b is a monic arrow $f: a \rightarrow b$.

Let $a_1, a_2 = \text{Ob } \mathcal{C}$ and $f_1: a_1 \rightarrow b, f_2: a_2 \rightarrow b$ pseudo-subobjects of b .

We say that $f_1 \subseteq f_2$, if there is $h: a_1 \rightarrow a_2$ s.t. $f_1 \text{ chr}$

$$\begin{array}{ccc} & a_2 & \\ & \uparrow & \searrow f_2 \\ h \uparrow & & b \\ a_1 & \xrightarrow{f_1} & \end{array}$$

ie, $f_1 = f_2 \circ h$

(Since the compositions f_1 and f_2 are monic, h or f_2 is also monic)

Proposition 2.1.2 Let f, g, κ pseudo-subobjects of $b = \text{Ob } \mathcal{C}$.

(i) $f \subseteq f$

(ii) $f \subseteq g \rightarrow g \subseteq \kappa \rightarrow f \subseteq \kappa$

Proof: (i)

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \text{id}_a \uparrow & & \\ a & \xrightarrow{f} & \end{array}$$

(ii)

$$\begin{array}{ccc} d & \xrightarrow{\kappa} & b \\ i \uparrow & & \\ c & \xrightarrow{g} & \\ h \uparrow & & \\ a & \xrightarrow{f} & \end{array}$$

$f = g \circ h, g = \kappa \circ i$

Then $\kappa \circ (i \circ h) = (\kappa \circ i) \circ h = g \circ h = f$

Remark 2.1.3

If f, g pseudo-subobjects of $b = \text{Ob } \mathcal{C}$ $A: f \subseteq g$ and $g \subseteq f$, then

if $f: a \rightarrow b$ and $g: c \rightarrow b$, we have that $a \cong c$

Proof:

$$\begin{array}{ccc} c & \xrightarrow{g} & b \\ i \uparrow \downarrow j & & \\ a & \xrightarrow{f} & \end{array}$$

$f = g \circ i, g = f \circ j$ (i, j monics too) We show $j \circ i = \text{id}_a$

$f = g \circ i = (f \circ j) \circ i = f \circ (j \circ i)$

$g = f \circ j = (g \circ i) \circ j = g \circ (i \circ j)$

Hence we have $f \circ (j \circ i) = f \circ 1_a (= f)$

Since f is monic, we conclude $j \circ i = 1_a$.

Similarly we have $g \circ (i \circ j) = g \circ 1_b (= g)$.

Since g is monic, we get $i \circ j = 1_b$. \square

• Definition 2.1.4

• We write $f \simeq g \iff f \leq g$ and $g \leq f$ ($\neq f=g$, just $a \approx c$)

and we say f, g are "isomorphic" pseudo-subobjects.

• Remark 2.1.5: \simeq is an equivalence relation

Proof: $f \simeq f \iff f \leq f$ and $f \leq f$.

• $f \simeq g \rightarrow g \simeq f$ ($f \leq g$ and $g \leq f \iff g \leq f$ and $f \leq g$)

• $f \leq g \rightarrow g \leq h \rightarrow f \leq h$: $f \leq g$ and $g \leq h$

\downarrow $\frac{g \leq h \text{ and } h \leq g}{h \leq g}$ \uparrow
 $\text{trans } f \leq h$ $h \leq f$

• Definition 2.1.6 If b obe, a subobject of b is an equivalence class $[f]_{\simeq} = \{g \mid f \simeq g\}$ of pseudo-subobjects of b .

If $[f]_{\simeq}, [g]_{\simeq}$ subobjects of b (or $\text{Sub}_e(b)$, or $\text{Sub}(b)$) we define

$[f]_{\simeq} \leq [g]_{\simeq}$ whenever $[f] \leq [g] \iff f \leq g$

• Corollary 2.1.7 If $[f], [g] \in \text{Sub}(b)$ s.t. $[f] \leq [g]$ and $[g] \leq [f]$, then $[f] = [g]$.

Proof: $[f] \leq [g] \iff f \leq g$, $[g] \leq [f] \iff g \leq f$. Hence $f \leq g$ and $g \leq f \implies f \simeq g \implies$

$[f] = [g]$.

• In this way we get a generalization of $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ then $A_1 = A_2$ (!)

• Usually we say the subobject f instead of $[f]$

$f \leq g \iff [f] \leq [g]$.

• Exercise: Show that in Set : $\mathcal{P}(B) \cong_{\text{Set}} \text{Sub}(B)$

Definition 2.18: Let \mathcal{C} be a category with 1 , and $a, o \in \text{Ob } \mathcal{C}$.

RECOVERY OF POINTS

An element of a is an arrow $x: 1 \rightarrow a$. (This arrow is always monic by Prop. 1.9.1)

$\text{El}(a) = \text{Ar}(1, a)$

Motivation If A is a set and $x \in A$, then $i_x: \{x\} \rightarrow A$ is an arrow.

If $f: 1 = \{*\} \rightarrow A$ is given, then $f(*) \in A$, the unique element of A determined by f .

Questions

- (i) a non-initial = Does a have elements? (Recall that if a is initial, $x: 1 \rightarrow a \rightarrow 0 \rightarrow 1$ for $x: 1 \rightarrow a$, and $0: 1 \rightarrow 0$, $x \circ 0 = 1$, and $0 \circ x = 0$, $x \circ 0 = 1$, i.e., $0 \cong 1$. If \mathcal{C} is cart. closed, then \mathcal{C} is degenerate)
- (ii) $\text{El}(a) = \text{El}(b) \rightarrow a = b$?
- (iii) f is monic $\stackrel{?}{\iff}$ for every $x, y \in \text{El}(a)$ $f(x) = f(y) \rightarrow x = y$?

also $1 \xrightarrow{x} a \xrightarrow{f} b$ $f(x) = f \circ x$

If f is monic $f(x) = f(y) \rightarrow x = y$.

What about the converse?

Remark 2.19

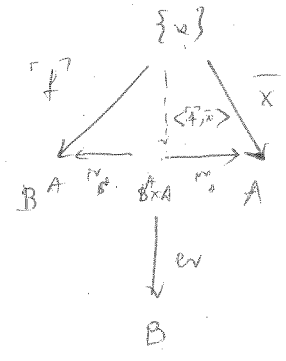
In Set if $f: A \rightarrow B$, then $f \in B^A$.

Let $\ulcorner f \urcorner = \{*\} \rightarrow B^A$ with $\ulcorner f \urcorner(*) = f$

If $x \in A$, then $\bar{x} = \{*\} \rightarrow A$, with $\bar{x}(*) = x$

$\ulcorner f \urcorner$ is the categorical element that corresponds to f , and \bar{x} is x .

Since $\text{ev}: B^A \times A \rightarrow B$ $\text{ev}(\ulcorner f \urcorner, \bar{x}) = f(x)$; by definition. Also, since Set is cartesian closed



$\text{pr}_{B^A} \circ \langle \ulcorner f \urcorner, \bar{x} \rangle = \ulcorner f \urcorner$

$\text{pr}_A \circ \langle \ulcorner f \urcorner, \bar{x} \rangle = \bar{x}$

$\langle \ulcorner f \urcorner, \bar{x} \rangle(*) \in B^A \times A$, and

$\langle \ulcorner f \urcorner, \bar{x} \rangle(*) = (\text{pr}_{B^A}(\langle \ulcorner f \urcorner, \bar{x} \rangle(*)), \text{pr}_A(\langle \ulcorner f \urcorner, \bar{x} \rangle(*)))$

$$= (\ulcorner f \urcorner(x), \bar{x}(x))$$

$$= (f, x)$$

Hence, $ev(\langle \ulcorner f \urcorner, \bar{x} \rangle(x)) = ev(f, x) = f(x)$

Moreover $f \circ \bar{x} : \{x\} \rightarrow B$ and

$$(f \circ \bar{x})(x) = f(\bar{x}(x)) = f(x)$$

i.e., $ev \circ \langle \ulcorner f \urcorner, \bar{x} \rangle = f \circ \bar{x}$

• Definition 2.1.10 Let \mathcal{C} be a cat with exponentials.

If $f: a \rightarrow b$

$pr_a: 1 \times a \rightarrow a$, then $f \circ pr_a: 1 \times a \rightarrow b$

We define the name $\ulcorner f \urcorner$ of f to be the exponential object $(c=1)$ of $f \circ pr_a$, i.e.,

$\ulcorner f \urcorner$ is the unique arrow $1 \rightarrow b^a$ st. $\ulcorner f \urcorner \circ pr_a = f$

$$\begin{array}{ccc}
 & b^a \times a & \\
 \ulcorner f \urcorner \times 1_a \uparrow & \searrow ev & \\
 1 \times a & \xrightarrow{f \circ pr_a} & b
 \end{array}
 \quad \text{is, } ev \circ (\ulcorner f \urcorner \times 1_a) = f \circ pr_a$$

• Corollary 2.1.11 If \mathcal{C} is a cat with exponentials, $f: a \rightarrow b$ and $x: 1 \rightarrow a$,

then

$$ev \circ \langle \ulcorner f \urcorner, x \rangle = f \circ x$$