Section 1.7 Limits and co-limits

Definition 1.7.1 • A diagram D in $\mathcal{C}$ is a collection of $\mathcal{C}$-objects $D = (d_i)_{i \in I}$ with some arrows $g(i,j) : d_i \to d_j$. (possibly more than one arrow $d_i \to d_j$, possibly none)

for simplicity we omit the arrows

• A cone for diagram D is $(c, (f_i)_{i \in I})$ where $c \in Ob\,\mathcal{C}$ and $f_i : c \to d_i$ $\forall i \in I$ s.t. tfdc

or $(f_i : c \to d_i)_{i \in I}$

$$\begin{tikzcd} d_i \arrow[rr, "g"] & & d_j \\ & c \arrow[ul, "f_i"] \arrow[ur, "f_j"'] & \end{tikzcd}$$

whenever g is ~~a diagram~~ an arrow in D.

• A limit for a diagram D is a D-cone $(f_i : c \to d_i)_{i \in I}$ s.t. if $(f_i' : c' \to d_i)_{i \in I}$ is any other D-cone, there is unique $f : c' \to c$ s.t. tfdc:

$$\begin{tikzcd} & d_i & \\ c' \arrow[ur, "f_i'"] \arrow[rr, "f"'] & & c \arrow[ul, "f_i"'] \end{tikzcd}$$

for every d_i in D. ("any other cone factors uniquely through the limit")

Remark 1.7.2 A limit for D is unique up to isomorphism.

Proof: (this proof is a "generalization" of the proof of the uniqueness of the product up to iso)

Let c' with the same universal property.

$$\begin{tikzcd} & d_i & \\ c \arrow[ur, "f_i"] \arrow[rr, "g"'] & & c' \arrow[ul, "f_i'"'] \end{tikzcd}$$

We have that

$$\begin{tikzcd} & & d_i & & \\ c' \arrow[urr, "f_i'"] \arrow[rr, "f"'] \arrow[rrrr, bend right, "1_{c'}"'] & & c \arrow[u, "f_i"] \arrow[rr, "g"'] & & c' \arrow[ull, "f_i'"'] \end{tikzcd}$$

$$f_i' \circ (g \circ f) = (f_i' \circ g) \circ f = f_i \circ f = f_i'$$

By uniqueness of the arrow between c' and c', and since $1_{c'}$ has this property too we get $g \circ f = 1_{c'}$

Similarly from the diagram

$$\begin{tikzcd} & & d_i & & \\ c \arrow[urr, "f_i"] \arrow[rr, "g"'] \arrow[rrrr, bend left=10, "1_c"] & & c' \arrow[u, "f_i'"] \arrow[rr, "f"'] & & c \arrow[ull, "f_i"'] \end{tikzcd}$$

we get $f \circ g = 1_c$. Hence $c \cong c'$. $\square$

Examples 1.7.3 (i) Let $D = [a, b]$ the diagram with no arrows. $\quad a \quad b$

A D-cone is $(c,\ f: c \to a,\ g: c \to b)$

$$\begin{tikzcd} a & & b \\ & c \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{tikzcd} a & & b \\ & c' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

A limit for D is a D-cone s.t. for every other D-cone $(c', f': c' \to a, g': c' \to b)$ there is unique e s.t.

$$\begin{tikzcd} & a & \\ c \arrow[ur, "\pi_a = f"] \arrow[dr, "\pi_b = g"'] & & c' \arrow[ll, "e"'] \arrow[ul, "f'"'] \arrow[dl, "g'"] \\ & b & \end{tikzcd}$$

$e \circ f' = f$ and $e \circ g' = g$

i.e. c is a product of a, b in $\mathcal{C}$.

(ii) $D = [a, b]$ and $f, g: a \to b$ i.e. $a \overset{f}{\underset{g}{\rightrightarrows}} b$ is our diagram.

A D-cone is $(c,\ h: c \to a,\ i: c \to b)$ s.t.

$$\begin{tikzcd} a \arrow[rr, "f"] & & b \\ & c \arrow[ul, "h"] \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, "g"] & & b \\ & c \arrow[ul, "h"] \arrow[ur, "i"'] & \end{tikzcd}$$

$i = f \circ h$, $\quad i = g \circ h$

i.e. $c \xrightarrow{h} a \overset{f}{\underset{g}{\rightrightarrows}} b \qquad f \circ h = g \circ h$

A D-limit is an equalizer.

(iii) $D = [\ ]$ the empty diagram

A D-cone is a $\mathcal{C}$-object c.

A limit for D is c s.t. for every other $c' \in Ob_{\mathcal{C}}$ there is $f: c' \to c$, i.e., c is terminal

Ex. Define co-cone for D

co-limits for D.

(i)', (ii)', (iii)' co-products, co-eq's, initial objects

• Section 1.8 Pullbacks and pushouts

Definition 1.8.1 Let the diagram D: $\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$ in $\mathcal{C}$.

A cone for D is $(d,\ g': d \to a,\ h: d \to c,\ f': d \to b)$

$$\begin{tikzcd} d \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, "h"] & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$$

s.t. $f \circ g' = h = g \circ f'$

Hence a D-cone is a pair $(d;\ g': d \to a,\ f': d \to b)$ s.t. the square commutes

$$\begin{tikzcd} d \arrow[r, "f'"] \arrow[d, "g'"'] & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$$

A pullback of $\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$ is a limit for D i.e., for every cone $(e,\ g'': e \to a,\ f'': e \to b)$ s.t. $g \circ f'' = f \circ g''$

there is unique $k: e \to d$ s.t. $f' \circ k = f''$ and $g' \circ k = g''$

$$\begin{tikzcd} e \arrow[dr, dashed, "k"] \arrow[drr, "f''"] \arrow[ddr, "g''"'] & & \\ & d \arrow[r, "f'"] \arrow[d, "g'"'] & b \arrow[d, "g"] \\ & a \arrow[r, "f"'] & c \end{tikzcd}$$

The inner square (f', g', f, g) is called pullback square (or Cartesian square)

We say that "f' arises by pulling back f along g"

"g' — " — " g — f".

• It is the most important limit in TT.

Examples 1.8.2 (i) In Set the pullback of $\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$

$$\begin{tikzcd} D \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is defined by $D = \{(x,y) \in A \times B \mid f(x) = g(y)\} = A \times_C B$: the prod. of A,B over C, or fibred product

$f': D \to B \quad f'(x,y) = y$

$g': D \to A \quad g'(x,y) = x$

Ex.: prove the universal property of D

$$\begin{tikzcd} E \arrow[dr, "x"] \arrow[drr, bend left, "f''"] \arrow[ddr, bend right, "g''"'] & & \\ & P \arrow[r, "f'"] \arrow[d, "g'"] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

$x: E \to P \subseteq A \times B$

$x(e) = (g''(e), f''(e))$, the only possibility from the data

x is well-defined; since

$$f(g''(e)) = g(f''(e))$$

and $f \circ g'' = g \circ f''$ is part of our hypothesis for x

Clearly $f'(x(e)) = f'(g''(e), f''(e))$
$= f''(e)$

$g'(x(e)) = g'(g''(e), f''(e))$
$= g''(e)$

By the last two conditions follows (apart from the def of x) the uniqueness of x. □

(ii) In Set, if $f: A \to B$ and $C \subseteq B$, then $f^{-1}(C) = \{x \in A \mid f(x) \in C\}$

$$\begin{tikzcd} f^{-1}(C) \arrow[r, "f'"] \arrow[d, hook, "i'"'] & C \arrow[d, hook, "i"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$A \times_B C = \{(x,y) \in A \times C \mid f(x) = i(y) = y\}$

$f'(x,y) = y = f(x)$

$i'(x,y) = x$.

Clearly: $A \times_B C \cong f^{-1}(C)$ $\quad (x,y) \in A \times_B C \overset{i'}{\mapsto} x$

$x \in f^{-1}(C) \mapsto (x,y)$, where y is the unique element of C s.t. $f(x) = y$.

Hence $f^{-1}(C)$ arises by pulling C back along f.

(dynamical character of the concept of function, we do not deal with sets of ordered pairs)

<u>Proposition 1.8.3</u> If $\begin{tikzcd} d \arrow[r, "i"] \arrow[d, "i"'] & a \arrow[d, "g"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$ is a pullback, then i is an equalizer of f, g

<u>Proof:</u>

$$\begin{tikzcd} e \arrow[dr, "x"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "h"'] & & \\ & d \arrow[r, "i"] \arrow[d, "i"'] & a \arrow[d, "g"] \\ & a \arrow[r, "f"'] & b \end{tikzcd}$$

can be written $\begin{tikzcd} d \arrow[r, "i"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\ e \arrow[u, dashed, "x"] \arrow[ur, "h"'] & & \end{tikzcd}$ i.e. $i = eq(f,g)$

$i \circ x = h$

<u>Proposition 1.8.4</u> If $\mathcal{C}$ has terminal 1 and $\begin{tikzcd} d \arrow[r, "g"] \arrow[d, "f"'] & b \arrow[d, "I_b"] \\ a \arrow[r, "I_a"'] & 1 \end{tikzcd}$ is a pullback, then d is a product of a, b

<u>Proof:</u>

$$\begin{tikzcd} e \arrow[dr, "x"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "f'"'] & & \\ & d \arrow[r, "g"] \arrow[d, "f"'] & b \arrow[d, "I_b"] \\ & a \arrow[r, "I_a"'] & 1 \end{tikzcd}$$

$I_b \circ g = I_a \circ f$, since $a \to 1$ is unique. Similarly

$I_b \circ g' = I_a \circ f'$. Hence the existence of x gives that d is a product.

Proposition 1.8.5 (The pullback lemma - PBL). Let the following diagram commute

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\ d \arrow[r, "e"] \arrow[d, "j"'] & c \arrow[d, "i"] \\ a' \arrow[r, "k"'] & b' \end{tikzcd}$$

(i) If the two small squares are pullbacks, then the outer rectangle $\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "j\circ h"'] & b \arrow[d, "i\circ g"] \\ a' \arrow[r, "k"'] & b' \end{tikzcd}$ is a pullback.

(ii) If the outer rectangle and the bottom square are pullbacks, then the top square is a pullback

(It is used in the proof of uniqueness of the sub-object classifier)

Proposition 1.8.6 $f: a \to b$ is monic $\Leftrightarrow$ $\begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "1_a"'] & a \arrow[d, "f"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$ is a pullback square

Proof: let $c \overset{g}{\underset{h}{\rightrightarrows}} a \xrightarrow{f} b$ st $f\circ g = f\circ h \to g = h$.

- Clearly $f\circ 1_a = f\circ 1_a = f$. Let c, h, g st $f\circ h = f\circ g$. By hypothesis $h = g$

Let

$$\begin{tikzcd} c \arrow[dr, dashed, "k"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\ & a \arrow[r, "1_a"] \arrow[d, "1_a"'] & a \arrow[d, "f"] \\ & a \arrow[r, "f"'] & b \end{tikzcd}$$

Define $k: c \to a$ $k = h$. Then $1_a \circ k = h = g$

If it is a pullback square then it's immediate that $h = 1_a \circ k = g$.

Proposition 1.8.7 If $\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "i"'] & b \arrow[d, "j"] \\ c \arrow[r, "g"'] & d \end{tikzcd}$ is a pullback square and g is monic, then f is monic

Proof of 1.5.7

(a)

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\
d \arrow[r, "e"] \arrow[d, "j"'] & c \arrow[d, "i"] \\
a' \arrow[r, "x"'] & b'
\end{tikzcd}$$

Hypothesis

$$\boxed{\begin{array}{ll} g \circ f = e \circ h & (1) \\ i \circ e = x \circ j & (2) \end{array}}$$

$$\begin{tikzcd}
a'' \arrow[dr, dashed, "x'"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "h'"'] & & \\
& a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\
& d \arrow[r, "e"'] & c
\end{tikzcd}$$

(I)

~~$f \circ g$~~ =

$g \circ f' = e \circ h'$

$\exists! x'$ ($f \circ x' = f'$ and $h \circ x' = h'$)

$$\begin{tikzcd}
d' \arrow[dr, dashed, "x''"] \arrow[drr, bend left, "e'"] \arrow[ddr, bend right, "j'"'] & & \\
& d \arrow[r, "e"] \arrow[d, "j"'] & c \arrow[d, "i"] \\
& a' \arrow[r, "x"'] & b'
\end{tikzcd}$$

$i \circ e' = x \circ j'$

$\exists! x''$ ($e \circ x'' = e'$ and $j \circ x'' = j'$) (II)

$$\begin{tikzcd}
c' \arrow[dr, dashed, "x_2"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "h'"'] & & \\
& a \arrow[r, "f"] \arrow[d, "j \circ h"'] & b \arrow[d, "i \circ g"] \\
& a' \arrow[r, "x"'] & b'
\end{tikzcd}$$

First we show that $(i \circ g) \circ f = x \circ (j \circ h)$

$$\begin{aligned}
(i \circ g) \circ f &= i \circ (g \circ f) \\
&\overset{(1)}{=} i \circ (e \circ h) \\
&= (i \circ e) \circ h \\
&\overset{(2)}{=} (x \circ j) \circ h \\
&= x \circ (j \circ h)
\end{aligned}$$

Let c' st. $(i \circ g) \circ g' = x \circ h'$
and h', g'

$i \circ (g \circ g') = x \circ h'$

$$\begin{tikzcd}
c' \arrow[dr, dashed, "x_1"] \arrow[drr, bend left, "g \circ g'"] \arrow[ddr, bend right, "h'"'] & & \\
& d \arrow[r, "e"] \arrow[d, "j"'] & c \arrow[d, "i"] \\
& a' \arrow[r, "x"'] & b'
\end{tikzcd}$$

By (II) there is $!x_1 : c' \to d$ st $e \circ x_1 = g \circ g'$
$j \circ x_1 = h'$

$$\begin{tikzcd}
c' \arrow[dr, dashed, "x_2"] \arrow[drr, bend left, "g'"] \arrow[ddr, bend right, "x_1"'] & & \\
& a \arrow[r, "f"] \arrow[d, "h"'] & b \arrow[d, "g"] \\
& d \arrow[r, "e"'] & c
\end{tikzcd}$$

First we show that $g \circ g' = e \circ x_1$, which is just

By (I) there is $! \, x_2 : c' \to a$ st.

$f \circ x_2 = g'$ and $h \circ x_2 = x_1$

We need to show $f \circ x_2 = g'$, which is)

and $(j \circ h) \circ x_2 = h'$

$$(j \circ h) \circ u_2 = j \circ (h \circ u_2)$$
$$= j \circ u_1$$
$$= h' \text{, by the def of } u_1.$$

(ii) Similarly.

Proof of Prop. 1.1.7.

$$\begin{tikzcd}
a' \arrow[dr, "h"] \arrow[drr, bend left, "f \circ h'"] \arrow[ddr, bend right, "i \circ h"'] & & \\
& a \arrow[r, "f"] \arrow[d, "i"] & b \arrow[d, "j"] \\
& c \arrow[r, "g"'] & d
\end{tikzcd}$$

$f \circ h = f \circ h'$

$j \circ f = g \circ i$

$$\begin{aligned}
j \circ (f \circ h') &= j \circ (f \circ h) \\
&= (j \circ f) \circ h \\
&= (g \circ i) \circ h \\
&= g \circ (i \circ h)
\end{aligned}$$

$\exists! u : (f \circ u = f \circ h')$

$i \circ u = i \circ h$

Since $f \circ h' = f \circ h$

$f \circ h = f \circ h$ $\quad i \circ h = i \circ h$

$i \circ h' = i \circ h$?

$$\begin{aligned}
g \circ (i \circ h') &= (g \circ i) \circ h' = (j \circ f) \circ h' \\
&= j \circ (f \circ h') \\
&= j \circ (f \circ h) \\
&= (j \circ f) \circ h \\
&= (g \circ i) \circ h \\
&= g \circ (i \circ h)
\end{aligned}$$

$\overset{g \text{ monic}}{\Longrightarrow}$ $i \circ h' = i \circ h$.

Hence by uniqueness of u, $h = h'$

$\square$

Section 1.9 Completeness and co-completeness

Definition 1.9.1 $\mathcal{C}$ is <u>complete</u>, if every diagram D in $\mathcal{C}$ has a limit in $\mathcal{C}$

$\mathcal{C}$ is <u>co-complete</u>, if " " " " co-limit in $\mathcal{C}$.

$\mathcal{C}$ is <u>bi-complete</u>, if it is complete and co-complete

$\mathcal{C}$ is <u>finitely complete</u>, if every finite diagram D in $\mathcal{C}$ (i.e., D has finite number of objects and arrows between them) has a limit in D.

$\mathcal{C}$ is <u>finitely co-complete</u>, if every finite diagram D in $\mathcal{C}$ has a colimit in D

$\mathcal{C}$ is <u>finitely bi-complete</u>, if it is finitely complete and finitely co-complete.

Theorem 1.9.2 If $\mathcal{C}$ has a terminal object 1 and a pullback for every pair of $\mathcal{C}$-arrows with common codomain $\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$, then $\mathcal{C}$ is finitely complete.

(+ dual of Th. 1.9.2) (...)

i.e., finite completeness rests only on pullbacks (that's why these are important), when a terminal object exists

Proof: ~~...~~ not complete. 1 and pullback $\Longrightarrow a \times b$ (Prop. 1.8.9)

$$\begin{tikzcd} d \arrow[r, "g"] \arrow[d, "f"'] & b \arrow[d] \\ a \arrow[r] & 1 \end{tikzcd}$$

pullbacks + products $\Longrightarrow$ equalizers (see Prop. 1.P.3)

(*)

Remark 1.9.3 The converse of Th. 1.9.2 holds trivially: the terminal object is the limit of the empty diagram and pullbacks were defined as limits.

(*) **Elaboration**: Given pullbacks + products (which we get from the previous step)

and if $a \underset{g}{\overset{f}{\rightrightarrows}} b$ are given, we get $a \underset{\langle 1_a, g\rangle}{\overset{\langle 1_a, f\rangle}{\rightrightarrows}} a \times b$ or the corner

$$\begin{tikzcd} & a \arrow[dl, "1_a"'] \arrow[d, dashed, "\langle 1_a, f\rangle"] \arrow[dr, "f"] & \\ a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dl, "1_a"'] \arrow[d, dashed, "\langle 1_a, g\rangle"] \arrow[dr, "g"] & \\ a & a \times b \arrow[l, "pr_a"] \arrow[r, "pr_b"'] & b \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[d, "\langle 1_a, g\rangle"] \\ a \arrow[r, "\langle 1_a, f\rangle"'] & a \times b \end{tikzcd}$$

Hence we get their pullback

$$\begin{tikzcd} d \arrow[r, "q"] \arrow[d, "p"'] & a \arrow[d, "\langle 1_a, g\rangle"] \\ a \arrow[r, "\langle 1_a, f\rangle"'] & a \times b \end{tikzcd}$$

Since the previous square commutes we have

$$\langle 1_a, g\rangle \circ q = \langle 1_a, f\rangle \circ p$$

But by the obvious properties (given exs in the section of products)

$$\left.\begin{array}{l}\langle 1_a, g\rangle \circ q = \langle 1_a \circ q, g \circ q\rangle = \langle q, g\circ q\rangle \\ \langle 1_a, f\rangle \circ p = \langle 1_a \circ p, f\circ p\rangle = \langle p, f\circ p\rangle\end{array}\right] \Rightarrow p = q \text{ and } f\circ p = g\circ q$$

Hence the pullback is written as

$$\begin{tikzcd} d \arrow[r, "p"] \arrow[d, "p"'] & a \arrow[d, "\langle 1_a, g\rangle"] \\ a \arrow[r, "\langle 1_a, f\rangle"'] & a\times b \end{tikzcd}$$

and by Prop. 1.8.3, $d \xrightarrow{p} a$ is an equalizer of $\langle 1_a, g\rangle$ and $\langle 1_a, f\rangle$

(If $\begin{tikzcd} d \arrow[r, "i"] \arrow[d, "i"'] & a \arrow[d, "g"] \\ a \arrow[r, "f"'] & b \end{tikzcd}$ is pullback, then $\mathrm{eq}(f, g)$.)

$$\begin{tikzcd} d \arrow[r, "p"] & a \arrow[r, shift left, "\langle 1_a, f\rangle"] \arrow[r, shift right, "\langle 1_a, g\rangle"'] & a\times b \\ c \arrow[u, dashed, "x"] \arrow[ur, "h"'] & & \end{tikzcd}$$

Since $\langle 1_a, f\rangle \circ p = \langle 1_a, g\rangle \circ p \Leftrightarrow f\circ p = g \circ p$

and similarly for h, x,

we get that $d \xrightarrow{p} a$ is also an equalizer of f, g. $\square$

- Remark in Makkai 8?, p. 37: If $\mathcal{C}$ is finitely complete, then the slice $\mathcal{C}/a$ is finitely complete but if $\mathcal{C}$ has exponentials, then $\mathcal{C}/a$ may not (that they do is an essential property of l.c.c.c., i.e. that's why we put this into the definition)

Section 1.10: Exponentials

Definition 1.10.1 In Set, if A, B : Set, then $B^A = \{f : A \to B\}$.

We define $ev : B^A \times A \to B$

$$ev(f, x) = f(x),$$

which is called the evaluation function w.r.t. A, B (or $ev_{A,B}$)

Proposition 1.10.2: For every C : Set and $g : C \times A \to B$

there is a unique $\hat{g} : C \to B^A$ s.t.

$$\begin{tikzcd} B^A \times A \arrow[dr, "ev"] & \\ & B \\ C \times A \arrow[uu, dashed, "\hat{g} \times id_A"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, where $(\hat{g} \times id_A)(c, x) = (\hat{g}(c), id_A(x)) = (\hat{g}(c), x)$, $c \in C$, $x \in A$

Proof: Let C : Set and $g : C \times A \to B$. From these data the only candidate $C \to B^A$ is

$$\hat{g} : C \to B^A$$
$$c \mapsto \hat{g}(c)$$
$$\hat{g}(c) : A \to B$$
$$\hat{g}(c)(x) \equiv g(c, x)$$

($\exists$) Clearly, $ev(\hat{g} \times id_A (c, x)) = ev(\hat{g}(c), x) = \hat{g}(c)(x) = g(c, x)$

(!) If $h : C \to B^A$ s.t. $ev(\hat{h} \times id_A (c, x)) = ev(\hat{h}(c), x) = \hat{h}(c)(x) = g(c, x)$

then $\hat{h}(c) = \hat{g}(c)$, $\forall c \in C$.

Definition 1.10.3 A $\mathcal{C}$ with products has exponentials if for every a, b : $Ob_{\mathcal{C}}$ there is b^a : $Ob_{\mathcal{C}}$ and $ev : b^a \times a \to b$, an evaluation arrow (or $ev_{a,b}$), s.t. for every c : $Ob_{\mathcal{C}}$ and $g : c \times a \to b$ there is <u>unique</u> $\hat{g} : c \to b^a$ s.t. t.f.d.c.

$$\begin{tikzcd} b^a \times a \arrow[dr, "ev"] & \\ & b \\ c \times a \arrow[uu, dashed, "\hat{g} \times 1_a"] \arrow[ur, "g"'] & \end{tikzcd}$$

$f : a \to b$, $g : c \to d$.
(recall $f \times g : a \times c \to b \times d$
$f \times g = \langle f \circ pr_a, g \circ pr_c \rangle$)

i.e., $ev \circ (\hat{g} \times 1_a) = g$.

$\left(b^a,\ ev : b^a \times a \to b\right)$ (Ex) · Uniqueness of exponential up to isomorphism.

$\forall c \in ob_{\mathcal{C}} \quad \forall g : c \times a \to b \quad \exists!\ \hat{g} : c \to b^a$ s.t. (I)

$$\begin{tikzcd} b^a \times a \arrow[dr, "ev"] & \\ & b \\ c \times a \arrow[uu, "\hat{g} \times 1_a"] \arrow[ur, "g"'] & \end{tikzcd} \qquad ev \circ (\hat{g} \times 1_a) = g$$

Let $(d,\ ev' : d \times a \to b)$

$\forall c \in ob_{\mathcal{C}} \quad \forall g' : c \times a \to b \quad \exists!\ \hat{g}' : c \to d$ (II)

$$\begin{tikzcd} d \times a \arrow[dr, "ev'"] & \\ & b \\ c \times a \arrow[uu, "\hat{g} \times 1_a"] \arrow[ur, "g'"'] & \end{tikzcd} \qquad ev' \circ (\hat{g}' \times 1_a) = g'$$

WTS want $f : d \rightleftarrows b^a$ iso

From (I) take $c = d$

$g = ev' : d \times a \to b$

$\exists!\ \hat{ev'} : d \to b^a$ st.

$$\begin{tikzcd} b^a \times a \arrow[dr, "ev"] & \\ & b \\ d \times a \arrow[uu, "\hat{ev'} \times 1_a"] \arrow[ur, "ev'"] & \\ & \\ b^a \times a \arrow[uu, "\hat{ev} \times 1_a"] \arrow[uuur, "ev"'] & \end{tikzcd}$$

outer diagram commutes

$ev = \cancel{(\hat{ev'} \times 1_a)}\, ev' \circ (\hat{ev} \times 1_a)$

$= ev \circ (\hat{ev'} \times 1_a) \circ (\hat{ev} \times 1_a)$

$= ev \circ \left[(\hat{ev'} \circ \hat{ev}) \times 1_a\right]$

Hence $\hat{ev'} \circ \hat{ev} = 1_{b^a}$

Similarly $\hat{ev} \circ \hat{ev'} = 1_d$

From (II) take $c = b^a$

$g' = ev : b^a \times a \to b$

$\exists!\ \hat{ev} : b^a \to d$ st.

$$\begin{tikzcd} d \times a \arrow[dr, "ev'"] & \\ & b \\ b^a \times a \arrow[uu, "\hat{ev} \times 1_a"] \arrow[ur, "ev"'] & \end{tikzcd}$$

$$d \xrightarrow{\hat{ev'}} b^a \xrightarrow{\hat{ev}} d$$

It suffices to show that $\hat{ev} \circ \hat{ev'}$ has the univ. prop. of ex for $c = d$ and $g' = ev'$, because then $\hat{ev} \circ \hat{ev'} = 1_d$, from !

$\cancel{(\hat{ev} \circ \hat{ev'}) \times 1_a}\ (\hat{ev'} \times 1_a) \circ (\hat{ev} \times 1_a) \overset{Ex}{=} (\hat{ev'} \circ \hat{ev}) \times (1_a \circ 1_a)$

$= (\hat{ev'} \circ \hat{ev}) \times 1_a$

i.e.

$$\begin{tikzcd} d \times a \arrow[dr, "ev'"] & \\ & b \\ d \times a \arrow[uu, "1_d \times 1_a"] \arrow[ur, "ev'"'] & \end{tikzcd}$$

~~Hence~~

Proposition 1.10.4 The function $\hat{}$: $Ar_{\mathscr{C}}(c\times a, b) \to Ar_{\mathscr{C}}(c, b^a)$ is a bijection $\quad g \mapsto \hat{g}$

Proof: · Let $g, h : c\times a \to b$ s.t. $\hat{g} = \hat{h}$. Then $\hat{g}\times 1_a = \hat{h}\times 1_a$, and $ev\circ(\hat{g}\times 1_a) = ev\circ(\hat{h}\times 1_a) \Leftrightarrow g = h$

· Let $\varphi : c \to b^a$

Hence $ev\circ(\varphi\times 1_a) : c\times a \to b$

We define $g = ev\circ(\varphi\times 1_a)$

Also, $g = ev\circ(\hat{g}\times 1_a)$

By the uniqueness of $\hat{g}$ we get $\hat{g} = \varphi$, i.e. $\hat{}$ is a surjection.

(g, $\hat{g}$ are called <u>exponential adjoints</u> of each other)

Definition 1.10.5 A cartesian closed category is a finitely complete category with exponentiation.

Eg. Set

Theorem 1.10.6 Let $\mathscr{C}$ be a cartesian closed cat with an initial object 0

(i) $0 \cong 0\times a$, a : obj

(ii) If $f : a \to 0$, then $a \cong 0$

(iii) If $0 \cong 1$, then all $\mathscr{C}$-objects are isomorphic ($\mathscr{C}$ is <u>degenerate</u>)

(iv) If $f : 0 \to a$, then f is monic (actually: The unique 0_a is monic)

(v) $a^1 \cong a$, $a^0 \cong 1$, $1^a \cong 1$ (where 1 is the terminal object which exists by the hypothesis of finitely complete)

Proof (i) By Prop 1.10.4 $\quad Ar_{\mathscr{C}}(0\times a, b) \cong Ar_{\mathscr{C}}(0, b^a)$

$$= [0_{b^a}] \quad \text{by initiality of } 0$$

I.e for every b, there is a unique arrow : $0\times a \to b$

I.e, $0\times a$ is initial in $\mathscr{C}$, hence $0\times a \cong 0$. (by Prop. 1.4.2)

(ii) We show that $a \cong 0\times a$ hence by (i) $a \cong 0$.

By the univ property of the product there is unique x s.t.

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[d, "x"] \arrow[dr, "1_a"] & \\ 0 & 0\times a \arrow[l, "pr_0"] \arrow[r, "pr_a"'] & a \end{tikzcd}$$

$pr_a \circ x = 1_a$

$pr_0 \circ x = f$

We need to show that $x \circ pr_a = 1_{0\times a}$

Since $x \circ pr_a : 0\times a \to 0\times a$, and by (i) $0\times a$ is initial, there is a unique arrow $0\times a \to 0\times a$, hence $x \circ pr_a = 1_{0\times a}$. ■

(iii) Let $\varphi : 1 \to 0$. Since $a \xrightarrow{!_a} 1 \xrightarrow{\varphi} 0$, $\varphi \circ !_a : a \to 0$, we use (ii) to get $a \cong 0$, for every a. $a \cong 0 \cong b \Rightarrow a \cong b$.

(iv) Let $b \overset{f}{\underset{g}{\rightrightarrows}} 0 \xrightarrow{0_a} a$ and $0_a \circ f = 0_a \circ g$

Since there is an arrow $b \to 0$, by (ii) $b \cong 0$ and b is initial. Hence there is unique arrow $b \to 0$, i.e., $f = g$.

(v) <u>Exercise</u>.

<u>Remark 1.10.7</u> In Set monic epics are iso (not true in <u>Mon</u>), but true in Grp, but Grp not cartesian closed, since $0 \cong 1$ but not degenerate

$\varphi : \{0\} \to G \quad \varphi(0) = e$

$\vartheta : G \to \{0\} \quad \vartheta(g) = 0, \ \forall g \in G$

not $G_1, G_2 : Grp \quad G_1 \cong G_2$

There are cart-closed cats with monic epics that are not iso ([illegible])

So we need to find extra property(ies) that characterise Set. (the subobject classifier).

Chapter II Basic Theory of Topoi

$\mathcal{P}(B) = \{A \subseteq B\}$.

$(\mathcal{P}(B), \subseteq)$ poset, hence a cat

where $A_1 \xrightarrow{f} A_2 \Leftrightarrow A_1 \subseteq A_2$ and ttdc

$$\begin{tikzcd} A_2 \arrow[dr, hook] & \\ & B \\ A_1 \arrow[uu] \arrow[ur, hook] & \end{tikzcd}$$

Section 2.1 Subobjects

Motivation:

$A \subseteq B$ then $id_A : A \hookrightarrow B$ is injective ($\Rightarrow$ monic).

If $f : A \rightarrowtail B$, then $Im(f) = \{f(x) \mid x \in A\} \subseteq B$. Actually $A \cong Im(f)$

Definition 2.1.1 Let $b : Ob_\mathcal{C}$. A pseudo-subobject of b is a monic arrow $f : a \rightarrowtail b$.

Let $a_1, a_2 : Ob_\mathcal{C}$ and $f_1 : a_1 \rightarrowtail b$, $f_2 : a_2 \rightarrowtail b$ pseudo-subobjects of b.

We say that $f_1 \subseteq f_2$, if there is $h : a_1 \to a_2$ s.t. ttdc

$$\begin{tikzcd} a_2 \arrow[dr, tail, "f_2"] & \\ & b \\ a_1 \arrow[uu, "h"] \arrow[ur, tail, "f_1"'] & \end{tikzcd}$$

i.e., $f_1 = f_2 \circ h$

(Since the composition f_1, and f_2 are monic, by an Ex. h is also monic)

Proposition 2.1.2 Let f, g, κ pseudo-subobjects of $b : Ob_\mathcal{C}$.

(i) $f \subseteq f$

(ii) $f \subseteq g \to g \subseteq \kappa \to f \subseteq \kappa$

Proof: (i)

$$\begin{tikzcd} a \arrow[dr, tail, "f"] & \\ & b \\ a \arrow[uu, "1_a"] \arrow[ur, tail, "f"'] & \end{tikzcd}$$

(ii)

$$\begin{tikzcd} d \arrow[ddr, tail, "\kappa"] & \\ & \\ c \arrow[uu, "i"] \arrow[r, tail, "g"] & b \\ & \\ a \arrow[uu, "h"] \arrow[uur, tail, "f"'] & \end{tikzcd}$$

$f = g \circ h$, $g = \kappa \circ i$

Then $\kappa \circ (i \circ h) = (\kappa \circ i) \circ h = g \circ h = f$

Remark 2.1.3 If f, g pseudo-subobjects of $b : Ob_\mathcal{C}$ w. $f \subseteq g$ and $g \subseteq f$, Then

if $f : a \rightarrowtail b$ and $g : c \rightarrowtail b$, we have that $a \cong c$

Proof:

$$\begin{tikzcd} c \arrow[dr, tail, "g"] \arrow[dd, shift left, "j"] & \\ & b \\ a \arrow[uu, shift left, "i"] \arrow[ur, tail, "f"'] & \end{tikzcd}$$

$f = g \circ i$ $g = f \circ j$ (i, j monics too) We show $j \circ i = 1_a$

$f = g \circ i = (f \circ j) \circ i = f \circ (j \circ i)$

$g = f \circ j = (g \circ i) \circ j = g \circ (i \circ j)$

Hence we have $f \circ (j \circ i) = f \circ 1_a \quad (= f)$

Since f is monic, we conclude $j \circ i = 1_a$.

Similarly we have $g \circ (i \circ j) = g \circ 1_b \quad (= g)$.

Since g is monic, we get $i \circ j = 1_b$. $\square$

· Definition 2.1.4
· We write $f \simeq g :\Leftrightarrow f \leq g$ and $g \leq f$ ($\not\Rightarrow f = g$, just $a \cong c$)

and we say f, g are "isomorphic" pseudo-subobjects.

· Remark 2.1.5: $\simeq$ is an equivalence relation

Proof: · $f \simeq f \Leftrightarrow f \leq g$ and $f \leq f$.

· $f \simeq g \to g \simeq f$ ($f \simeq g \Leftrightarrow f \leq g$ and $g \leq f \Leftrightarrow g \leq f$ and $f \leq g \Leftrightarrow g \simeq f$)

· $f \simeq g \to g \simeq h \to f \simeq h$: $f \leq g$ and $g \leq f$

$$\frac{g \leq h}{\text{trans } f \leq h} \quad \text{and} \quad \frac{h \leq g}{h \leq f}$$

· Definition 2.1.6 If $b : Ob_{\mathcal{C}}$, a subobject of b is an equivalence class $[f]_{\simeq} = \{g \mid f \simeq g\}$ of pseudo-subobjects of b.

If $[f]_{\simeq}, [g]_{\simeq}$ subobjects of b (or: $\mathrm{Sub}_{\mathcal{C}}(b)$, or $\mathrm{Sub}(b)$) we define

$[f]_{\simeq} \subseteq [g]_{\simeq}$ or simpler $[f] \leq [g] :\Leftrightarrow f \leq g$

· Corollary 2.1.7 If $[f], [g] : \mathrm{Sub}(b)$ s.t. $[f] \leq [g]$ and $[g] \leq [f]$, then $[f] = [g]$.

Proof: $[f] \leq [g] \Leftrightarrow f \leq g$, $[g] \leq [f] \Leftrightarrow g \leq f$. Hence $f \leq g$ and $g \leq f \Rightarrow f \simeq g \Rightarrow$

$[f] = [g]$.

· In this way we get a generalization of $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ then $A_1 = A_2$. (!)

· Usually we say the subobject f instead of $[f]$

—//— $f \leq g$ —//— $[f] \leq [g]$.

· Exercise: Show that in Set, $\mathcal{P}(B) \cong \mathrm{Sub}_{Set}(B)$

RECOVERY OF ABSTRACT POINTS

Definition 2.1.8: Let $\mathcal{C}$ be a cat with 1, and $a \in Ob$.
An <u>element</u> of a is an arrow $x : 1 \to a$. (this arrow is always monic by Prop. 1.4.7)
$El(a) = Arr(1, a)$

Motivation If A = Set and $x \in A$, then $i : \{x\} \to A$ is an arrow.
If $f : 1 = \{*\} \to A$ is given, then $f(*) \in A$, the unique element of A determined by f.

Questions (i) a non-initial : Does a have elements? (Recall that if a is initial, and $x : 1 \to 0 \xrightarrow{0_1} 1$ then ~~1_1~~ $0_1 \circ x = 1_1$, and $0 \xrightarrow{0_1} 1 \xrightarrow{x} 0$ $x \circ 0_1 = 1_0$, i.e., $0 \cong 1$. If $\mathcal{C}$ is Cart. closed, then $\mathcal{C}$ is degenerate)

(ii) $El(a) = El(b) \to a = b$?

(iii) f is monic $\overset{?}{\Leftrightarrow}$ for every $x, y \in El(a)$ $f(x) = f(y) \to x = y$?
$f : a \to b$

also ~~1~~ $1 \underset{y}{\overset{x}{\rightrightarrows}} a \xrightarrow{f} b$ $\qquad f(x) = f \circ x$

If f is monic $f(x) = f(y) \to x = y$.

What about the converse?

Remark 2.1.9 In Set if $f : A \to B$, then $f \in B^A$.
Let $\ulcorner f \urcorner : \{*\} \to B^A$ with $\ulcorner f \urcorner (*) = f$
If $x \in A$, then $\overline{x} : \{*\} \to A$, with $\overline{x}(*) = x$
$\ulcorner f \urcorner$ is the categorical element in B^A that corresponds to f, and
$\overline{x}$ is " " x.

Since $ev : B^A \times A \to B$ $\quad ev((f, x)) = f(x)$, by definition. Also, since Set is cartesian closed

$$\begin{tikzcd}
& \{*\} \arrow[dl, "\ulcorner f \urcorner"'] \arrow[d, dashed, "\langle \ulcorner f \urcorner, \overline{x} \rangle"] \arrow[dr, "\overline{x}"] & \\
B^A & B^A \times A \arrow[l, "pr_{B^A}"'] \arrow[r, "pr_A"] \arrow[d, "ev"] & A \\
& B &
\end{tikzcd}$$

$pr_{B^A} \circ \langle \ulcorner f \urcorner, \overline{x} \rangle = \ulcorner f \urcorner$

$pr_A \circ \langle \ulcorner f \urcorner, \overline{x} \rangle = \overline{x}$

$\langle \ulcorner f \urcorner, \overline{x} \rangle (*) \in B^A \times A$, and

$\langle \ulcorner f \urcorner, \overline{x} \rangle (*) = \Big(pr_{B^A} \big(\langle \ulcorner f \urcorner, \overline{x} \rangle (*)\big), \ pr_A \big(\langle \ulcorner f \urcorner, \overline{x} \rangle (*)\big) \Big)$

$$= \left(\ulcorner f \urcorner (*), \ \bar{x}(*) \right)$$

$$= (f, x)$$

Hence $\quad ev\left(\langle \ulcorner f \urcorner, \bar{x} \rangle (*) \right) = ev(f, x) = f(x)$

Moreover $\quad f \circ \bar{x} : \{*\} \longrightarrow B$, and

$$(f \circ \bar{x})(*) = f(\bar{x}(*)) = f(x)$$

i.e., $\boxed{ev \circ \langle \ulcorner f \urcorner, \bar{x} \rangle = f \circ \bar{x}}$

- <u>Definition 2.1.10</u> Let $\mathcal{C}$ be a cat with exponentials.

  If $f : a \longrightarrow b$

  $pr_a : 1 \times a \longrightarrow a$, then $f \circ pr_a : 1 \times a \longrightarrow b$

We define the <u>name</u> $\ulcorner f \urcorner$ of f to be the exponential adjoint $(c=1)$ of $f \circ pr_a$, i.e.,

$\ulcorner f \urcorner$ is the unique arrow $: 1 \longrightarrow b^a$ s.t. the f.d.c.

$$\begin{tikzcd}
b^a \times a \arrow[dr, "ev"] & \\
& b \\
1 \times a \arrow[uu, "\ulcorner f \urcorner \times 1_a"] \arrow[ur, "f \circ pr_a"'] &
\end{tikzcd}$$

i.e., $ev \circ (\ulcorner f \urcorner \times 1_a) = f \circ pr_a$

- <u>Corollary 2.1.11</u> If $\mathcal{C}$ is a cat with exponentials, $f : a \longrightarrow b$ and $x : 1 \longrightarrow a$,

  then $\boxed{ev \circ \langle \ulcorner f \urcorner, x \rangle = f \circ x}$